Intro To Cosmology: Master Review Guide

Ansh Thukral

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2 Constants, Units, and Benchmark Values

2.1 Fundamental constants

$$c = 3.00 \times 10^{8}\ \mathrm{m\,s^{-1}}$$

$$G = 6.67 \times 10^{-11}\ \mathrm{m^{3}\,kg^{-1}\,s^{-2}}$$

$$k = 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} = 8.617 \times 10^{-5}\ \mathrm{eV\,K^{-1}}$$

$$h = 6.626 \times 10^{-34}\ \mathrm{J\,s}$$

$$\hbar = \frac{h}{2\pi}$$

$$\sigma = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$$

2.2 Astrophysical units

$$1\ \mathrm{AU} = 1.50 \times 10^{11}\ \mathrm{m}$$

$$1\ \mathrm{pc} = 3.09 \times 10^{16}\ \mathrm{m}$$

$$1\ \mathrm{Mpc} = 3.09 \times 10^{22}\ \mathrm{m}$$

$$1\ \mathrm{yr} = 3.16 \times 10^{7}\ \mathrm{s}$$

$$1\ \mathrm{Gyr} = 3.16 \times 10^{16}\ \mathrm{s}$$

$$M_{\odot} = 1.99 \times 10^{30}\ \mathrm{kg}$$

$$L_{\odot} \approx 3.83 \times 10^{26}\ \mathrm{W}$$

2.3 Particle and energy units

$$1\ \mathrm{eV} = 1.60 \times 10^{-19}\ \mathrm{J}$$

$$1\ \mathrm{MeV} = 1.60 \times 10^{-13}\ \mathrm{J}$$

$$m_e c^2 = 0.511\ \mathrm{MeV}$$

$$m_p c^2 \approx 938.3\ \mathrm{MeV}$$

$$m_n c^2 \approx 939.6\ \mathrm{MeV}$$

2.4 Benchmark cosmological values

Using the course benchmark values:

$$H_0 \approx 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$$

$$t_H = H_0^{-1} \approx 14.4\ \mathrm{Gyr}$$

$$d_H = \frac{c}{H_0} \approx 4.4\ \mathrm{Gpc}$$

For the cosmic microwave background:

$$T_0 = 2.7255\ \mathrm{K}$$

$$n_\gamma \approx 411\ \mathrm{cm^{-3}}$$

$$\eta \equiv \frac{n_{\mathrm{bary}}}{n_\gamma} \approx 6 \times 10^{-10}$$

For recombination / decoupling:

$$T_{\mathrm{rec}} \approx 3760\ \mathrm{K}$$

$$z_{\mathrm{rec}} \approx 1380$$

$$z_{\mathrm{dec}} \approx 1090$$

For nucleosynthesis:

$$kT_{\mathrm{freeze}} \approx 0.8\ \mathrm{MeV}$$

$$T_{\mathrm{freeze}} \approx 9 \times 10^9\ \mathrm{K}$$

$$t_{\mathrm{freeze}} \sim 1\ \mathrm{s}$$

$$T_{\mathrm{nuc}} \approx 7.6 \times 10^8\ \mathrm{K}$$

$$t_{\mathrm{nuc}} \approx 200\ \mathrm{s}$$

$$\left(\frac{n_n}{n_p}\right)_{\mathrm{freeze}} \approx \frac{1}{5}$$

$$Y_p \approx 0.24$$

$$\frac{D}{H} \approx 2.53 \times 10^{-5}$$

For the benchmark late-time model:

$$\Omega_{r,0} \approx 9 \times 10^{-5}, \qquad \Omega_{m,0} \approx 0.31, \qquad \Omega_{\Lambda,0} \approx 0.69$$

3 Important Values, Relationships, and MCQ Traps

3.1 Benchmark values to know immediately

These values are frequently useful in short-answer or MCQ questions:

- Present-day scale factor:

$$a_0 = 1$$

- CMB temperature today:

$$T_0 = 2.7255\ \mathrm{K}$$

- Present photon density:

$$n_\gamma \approx 411\ \mathrm{cm^{-3}}$$

- Baryon-to-photon ratio:

$$\eta \approx 6 \times 10^{-10}$$

- Recombination:

$$T_{\mathrm{rec}} \approx 3760\ \mathrm{K}, \qquad z_{\mathrm{rec}} \approx 1380$$

- Decoupling / last scattering:

$$z_{\mathrm{dec}} \approx 1090$$

- Nucleosynthesis:

$$T_{\mathrm{nuc}} \approx 7.6 \times 10^8\ \mathrm{K}, \qquad t_{\mathrm{nuc}} \approx 200\ \mathrm{s}$$

- Neutron–proton freezeout:

$$T_{\mathrm{freeze}} \approx 9 \times 10^9\ \mathrm{K}, \qquad \left(\frac{n_n}{n_p}\right)_{\mathrm{freeze}} \approx \frac{1}{5}$$

- Primordial helium fraction:

$$Y_p \approx 0.24$$

3.2 High-yield relationships

$$1 + z = \frac{a_0}{a_{\mathrm{em}}} = \frac{\lambda_{\mathrm{obs}}}{\lambda_{\mathrm{em}}}$$

With the convention $a_0 = 1$,

$$a_{\mathrm{em}} = \frac{1}{1+z}$$

$$v = H_0 r \qquad \text{(low-}z\text{ Hubble law)}$$

$$d_L = (1+z)\, d_P$$

$$d_A = \frac{d_P}{1+z}$$

$$d_A = \frac{d_L}{(1+z)^2}$$

$$H = \frac{\dot{a}}{a}$$

$$\varepsilon \propto a^{-3(1+w)}$$

Important special cases:

$$w = 0 \Rightarrow \varepsilon_m \propto a^{-3}$$

$$w = \frac{1}{3} \Rightarrow \varepsilon_r \propto a^{-4}$$

$$w = -1 \Rightarrow \varepsilon_\Lambda = \text{constant}$$

For a flat one-component universe:

$$a(t) \propto t^{2/[3(1+w)]}$$

Curvature sign and density:

$$k = +1 \Rightarrow \Omega > 1$$

$$k = 0 \Rightarrow \Omega = 1$$

$$k = -1 \Rightarrow \Omega < 1$$

Curvature term in the Friedmann equation:

$$-\frac{kc^2}{R_0^2 a^2}$$

Thus curvature scales as

$$a^{-2}.$$

3.3 Conceptual distinctions frequently tested in MCQs

Isotropy vs homogeneity

- **Isotropy**: looks the same in every direction from a given point.
- **Homogeneity**: looks the same at every location.
- Isotropy about one point does **not** imply homogeneity.
- Isotropy about every point **does** imply homogeneity.

Copernican principle vs cosmological principle

- **Copernican principle**: our location is not special.
- **Cosmological principle**: the universe is homogeneous and isotropic on large scales.

Recombination vs decoupling

- **Recombination**: electrons and protons bind into neutral hydrogen.
- **Decoupling**: photons stop scattering efficiently and free-stream.
- These are related, but not identical.

Distances

- **Proper distance**: separation at a chosen cosmic time.
- **Luminosity distance**: used with flux and luminosity.
- **Angular-diameter distance**: used with physical size and angular size.

Matter, radiation, and Λ

- Matter dilutes as volume increases.
- Radiation also loses energy per photon due to redshift.
- Λ has positive energy density but negative pressure.

Inflation

- Inflation solves the flatness, horizon, and monopole problems.
- Reheating is required because inflation dilutes ordinary particles and radiation.
- Quantum fluctuations are stretched by inflation and seed structure formation.

3.4 Sign and convention traps

- $a_0 = 1$ is a normalization convention, not a measured number.
- Larger z means smaller scale factor at emission.
- Low-z formulas such as $z \approx v/c$ and $v = H_0 r$ should not be used blindly at large redshift.
- $k = -1$ means negatively curved / open spatial geometry, not negative energy density.
- The curvature term in Friedmann has a minus sign:

$$-\frac{kc^2}{R_0^2 a^2}.$$

- Apparent magnitude decreases when a source gets brighter.

3.5 One-line conceptual bullets

- The dark sky at night implies at least one assumption of the static infinite-universe picture is false.
- Dust cannot permanently solve Olbers paradox because absorbed starlight is reradiated.
- Hubble expansion does not imply we are at the center.
- Pressure contributes to gravity in general relativity.
- Radiation redshifts faster than matter because its number density decreases and each photon loses energy.
- The angular-diameter distance can have a maximum, so very distant objects can appear larger again.
- Recombination happens well below 13.6 eV because of the high-energy tail of the blackbody spectrum and the huge photon-to-baryon ratio.
- Big Bang nucleosynthesis is limited by the deuterium bottleneck and by the absence of stable nuclei at $A = 5$ and $A = 8$.
- Deuterium is the best baryometer because its primordial abundance depends strongly on η.

4 Core Formula Overview

4.1 Observational cosmology

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$$

$$1 + z = \frac{a_0}{a_{\text{em}}}$$

$$v \approx cz \qquad (z \ll 1)$$

$$v = H_0 r$$

$$t_H = \frac{1}{H_0}$$

$$d_H = \frac{c}{H_0}$$

4.2 Blackbody radiation and the CMB

$$E = hf$$

$$\varepsilon_\gamma = \alpha T^4$$

$$n_\gamma = \beta T^3$$

$$\langle E \rangle \approx 2.70\,kT$$

$$hf_{\text{peak}} \approx 2.82\,kT$$

$$P = \sigma A T^4$$

4.3 Geometry and relativity

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_k(r)^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right]$$

$$d_P(t) = a(t)\, r$$

For light,

$$ds = 0$$

4.4 Cosmic dynamics

$$H = \frac{\dot{a}}{a}$$
$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$
$$\dot{\varepsilon} = -3H(\varepsilon + P)$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda c^2}{3}$$
$$P = w\varepsilon$$
$$\varepsilon \propto a^{-3(1+w)}$$

4.5 Flat one-component universes

$$a(t) \propto t^{2/[3(1+w)]}$$

Important cases:

$$a(t) \propto t^{2/3} \qquad \text{matter-only}$$
$$a(t) \propto t^{1/2} \qquad \text{radiation-only}$$
$$a(t) \propto e^{Ht} \qquad \Lambda\text{-only}$$

4.6 Distance measures

$$d_P(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}$$
$$d_L = (1+z)d_P$$
$$d_A = \frac{d_P}{1+z} = \frac{d_L}{(1+z)^2}$$
$$m - M = 5\log_{10}\left(\frac{d_L}{10\text{ pc}}\right)$$
$$\delta\theta = \frac{\ell}{d_A}$$

4.7 Dark matter and lensing

$$M(R) = \frac{v^2 R}{G}$$
$$M \sim \frac{\langle v^2\rangle r_h}{\alpha G} \qquad \text{(virial estimate)}$$
$$\alpha = \frac{4GM}{bc^2} \qquad \text{(lensing deflection)}$$

4.8 Recombination and nucleosynthesis

Hydrogen recombination / Saha form:

$$\frac{X^2}{1-X} = S(T,\eta)$$

Neutron-to-proton equilibrium:

$$\frac{n_n}{n_p} = e^{-Q_n/kT}$$

Neutron survival after time t:

$$f = e^{-t/\tau_n}$$

Maximum helium fraction if all neutrons end in ^{4}He:

$$Y_{\max} = \frac{2f_n}{1+f_n}, \qquad f_n \equiv \frac{n_n}{n_p}$$

4.9 Inflation

Exponential expansion:

$$a(t) \propto e^{Ht}$$

Number of e-foldings:

$$N = \ln\left(\frac{a_f}{a_i}\right)$$

Before inflation, in a radiation-dominated universe:

$$|\Omega - 1| \propto t$$

Inflation dilutes relic number densities exponentially:

$$n \propto e^{-3N}$$

5 Chapter 1 — Introduction and Physical Scales

5.1 What cosmology studies

Cosmology studies the universe as a whole: its large-scale structure, contents, geometry, expansion history, and possible origin and fate. Unlike many other branches of physics, cosmology often cannot rely on controlled experiments. Instead, it combines:

- observational astronomy,
- general relativity,
- thermodynamics and statistical mechanics,
- nuclear and particle physics.

A central simplifying assumption in cosmology is that, on sufficiently large scales, the universe can be modeled as smooth and isotropic. Small-scale complexity such as stars, galaxies, and clusters is then treated as structure superimposed on a simple large-scale background.

5.2 Why idealized smooth models are used

The real universe is very inhomogeneous on small scales:

- planets are clumped into solar systems,
- stars are clumped into galaxies,
- galaxies are clumped into groups, clusters, filaments, and walls.

However, on large enough scales, the detailed lumpiness averages out. This motivates the use of idealized models in which the universe is treated as:

- **homogeneous**: the same at every location,
- **isotropic**: the same in every direction.

These assumptions lead to the Robertson–Walker metric and the Friedmann equations later in the course.

5.3 Hot Big Bang framework

The standard cosmological model is the **Hot Big Bang model**. Its core claims are:

- the universe was once much hotter and denser than it is now,
- it has been expanding and cooling with time,
- this thermal history explains recombination, the cosmic microwave background, and primordial nucleosynthesis.

The Hot Big Bang model does *not* necessarily mean an explosion into pre-existing space. Rather, it describes the expansion of space itself from an earlier hot, dense state.

5.4 Important units and scales

Cosmology uses a mixture of astrophysical and particle-physics units.

Distance units

$$1 \text{ AU} = 1.50 \times 10^{11} \text{ m}$$

$$1 \text{ pc} = 3.09 \times 10^{16} \text{ m}$$

$$1 \text{ Mpc} = 3.09 \times 10^{22} \text{ m}$$

Typical uses:
- AU: solar-system scales
- pc: stellar distances
- Mpc: galactic and cosmological distances

Time units

$$1 \text{ yr} = 3.16 \times 10^{7} \text{ s}, \qquad 1 \text{ Gyr} = 3.16 \times 10^{16} \text{ s}$$

Typical cosmological ages are measured in Gyr, while early-universe physics often uses seconds or much shorter times.

Mass and luminosity units

$$M_{\odot} = 1.99 \times 10^{30} \text{ kg}, \qquad L_{\odot} \approx 3.83 \times 10^{26} \text{ W}$$

These are convenient reference scales for stars and galaxies.

Particle / energy units

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$$

$$1 \text{ MeV} = 1.60 \times 10^{-13} \text{ J}$$

$$1 \text{ GeV} = 10^{3} \text{ MeV}$$

These are especially useful for:
- recombination (~ 10 eV),
- nucleosynthesis ($\sim$ MeV),
- early-universe particle processes.

5.5 Planck units and why they matter

By combining G, c, $\hbar$, and k, one can construct natural units associated with quantum gravity:
- Planck time,
- Planck length,
- Planck mass / energy,
- Planck temperature.

The **Planck time**

$$t_P \sim 5 \times 10^{-44} \text{ s}$$

marks a regime where both quantum effects and gravity are important, and our current theories are incomplete.

You do not usually need to derive Planck units on the exam, but you should know their physical significance: they represent the earliest regime where classical cosmology is unlikely to be reliable.

5.6 What counts as evidence in cosmology

Cosmological models are tested using observations such as:
- galaxy redshifts,
- the cosmic microwave background,
- light-element abundances,
- supernova distances,
- large-scale structure,
- gravitational lensing.

Thus cosmology is both highly theoretical and strongly observation-driven.

5.7 High-yield Chapter 1 reminders

- Cosmology studies the universe on the largest scales, not individual stars or planets.
- The universe is modeled as smooth only on sufficiently large scales.
- The Hot Big Bang model means the universe was once hotter and denser, then expanded and cooled.
- Cosmology constantly mixes astrophysical units and particle-physics units.
- Planck scales signal the breakdown of classical cosmology.

5.8 Typical Chapter 1 exam asks

- Explain why cosmologists use idealized smooth models.
- Distinguish astrophysical from particle-physics units and convert between them.
- State what the Hot Big Bang model claims.
- Explain the significance of Planck scales.

5.9 Chapter 1 traps / reminders

- Do not describe the Big Bang as an explosion into empty space unless the wording is made more careful.
- Do not confuse local inhomogeneity with large-scale homogeneity.
- Keep track of unit systems; many numerical mistakes come from conversion errors.

6 Chapter 2 — Fundamental Observations

6.1 Olbers paradox

Olbers paradox asks: if the universe were
- infinite,
- static,
- eternal,
- and uniformly filled with stars,

why is the night sky dark?

In such a universe, every line of sight would eventually intersect a star. Since surface brightness does not decrease with distance in the required way to save the argument, the sky should be bright everywhere.

Why the paradox matters

It shows that at least one assumption of the static eternal universe must be false.

Why dust does not solve it

Interstellar dust can absorb starlight temporarily, but then it heats up and reradiates. Thus dust cannot permanently remove the energy from the radiation field.

Main resolution

The crucial resolution is that the universe has a **finite age**. Light from arbitrarily distant stars has not had enough time to reach us.

6.2 Cosmological principle and Copernican principle

Copernican principle

We do not occupy a special place in the universe.

Cosmological principle

On sufficiently large scales, the universe is homogeneous and isotropic.

The Copernican principle plus the observation that the universe looks approximately isotropic around us motivates the cosmological principle.

6.3 Isotropy and homogeneity

- **Isotropy**: same in every direction from a given point.
- **Homogeneity**: same at every location.

Important logical distinctions:

- isotropy about one point does *not* imply homogeneity,
- isotropy about every point *does* imply homogeneity.

A useful mental example is a bullseye pattern: isotropic about the center, but not homogeneous.

6.4 Redshift

Redshift is defined by

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}.$$

Equivalently,

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}}.$$

In expanding-universe language,

$$1 + z = \frac{a_0}{a_{\text{em}}}.$$

With the standard normalization $a_0 = 1$,

$$a_{\text{em}} = \frac{1}{1+z}.$$

This is one of the most important relations in the course.

6.5 Hubble law

For nearby galaxies (small redshift),

$$v \approx cz,$$

and Hubble's law is

$$v = H_0 r.$$

Combining them at low z,

$$z \approx \frac{H_0}{c} r.$$

Interpretation

The Hubble law is what a homogeneous, isotropic expansion predicts. It does *not* imply that we are at the center of the universe. Every observer in such a universe sees distant galaxies receding.

6.6 Hubble time and Hubble distance

The **Hubble time** is

$$t_H = \frac{1}{H_0},$$

which gives a characteristic timescale for the age of the universe.

The **Hubble distance** is

$$d_H = \frac{c}{H_0},$$

which gives a characteristic cosmological length scale.

Using the course benchmark,

$$H_0 \approx 68 \text{ km s}^{-1} \text{ Mpc}^{-1},$$

so

$$t_H \approx 14.4 \text{ Gyr}, \qquad d_H \approx 4.4 \text{ Gpc}.$$

6.7 Matter and radiation content of the universe

The universe contains several distinct components:

- **baryons**: protons and neutrons, making up ordinary matter,
- **electrons**: important for charge neutrality, but negligible for mass budget,
- **photons**: especially the CMB,
- **neutrinos**: very abundant and weakly interacting,
- **dark matter**: massive and inferred gravitationally,
- later: Λ / dark energy.

Why electrons are often ignored in mass density

The universe is electrically neutral, so electrons are present in roughly the same number as protons, but

$$m_e \ll m_p.$$

Therefore electrons contribute little to the mass density compared with baryons.

6.8 Blackbody radiation

A blackbody spectrum is determined entirely by temperature.

Important relations:

$$E = hf$$

$$\varepsilon_\gamma = \alpha T^4$$

$$n_\gamma = \beta T^3$$

$$\langle E \rangle \approx 2.70\, kT$$

$$hf_{\text{peak}} \approx 2.82\, kT$$

The Stefan–Boltzmann law for emitted power is

$$P = \sigma A T^4.$$

6.9 Cosmic microwave background

The CMB is relic blackbody radiation from the early universe. Its present temperature is

$$T_0 = 2.7255 \text{ K}.$$

Its spectrum is extremely close to a perfect blackbody, which is one of the strongest pieces of evidence for a hot, dense early universe.

Why the CMB supports the Big Bang

A hot, dense, opaque early universe naturally produces thermal radiation. As the universe expands, the radiation cools while retaining its blackbody form. The observed CMB is precisely what this scenario predicts.

6.10 Why the universe is dark at night but filled with the CMB

The optical night sky is dark because the observable universe has finite age and the stars do not cover every line of sight. But the universe is nevertheless filled with low-temperature radiation from the early universe, namely the CMB.

6.11 High-yield Chapter 2 formulas

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}}$$

$$1 + z = \frac{a_0}{a_{\text{em}}}$$

$$v \approx cz \qquad (z \ll 1)$$

$$v = H_0 r$$

$$t_H = \frac{1}{H_0}$$

$$d_H = \frac{c}{H_0}$$

$$\varepsilon_\gamma = \alpha T^4$$

$$n_\gamma = \beta T^3$$

$$\langle E \rangle \approx 2.70\, kT$$

6.12 Typical Chapter 2 exam asks

- Explain Olbers paradox and why dust is not a complete solution.
- Distinguish isotropy from homogeneity.
- Use redshift to find wavelength shift or scale factor.
- Use Hubble's law to estimate distance or recession speed.
- Compute Hubble time or Hubble distance.
- Explain why the CMB supports the Big Bang.
- Use blackbody formulas to estimate photon energy, number density, or energy density.

6.13 Chapter 2 traps / reminders

- Do not use $z \approx v/c$ unless the problem is explicitly low-z.
- Do not say Hubble expansion means we are at the center.
- Do not confuse a blackbody photon-energy relation with the peak of the spectrum.
- Remember:
$$1 + z = \frac{a_0}{a_{\text{em}}},$$
not $z = a_0/a_{\text{em}}$.
- Isotropy and homogeneity are related but not identical concepts.

7 Chapter 3 — Newton vs Einstein

7.1 Newtonian gravity vs general relativity

In Newtonian gravity, gravity is a force acting instantaneously at a distance between masses. In general relativity, gravity is not treated as a force in the usual sense; instead, matter and energy curve spacetime, and freely moving objects follow geodesics in that curved spacetime.

Newtonian gravity is adequate for many weak-field, low-speed problems, but cosmology requires a relativistic description because:

- the universe evolves dynamically,
- light propagation matters,
- pressure contributes to gravity,
- spacetime curvature is essential.

7.2 Special relativity essentials

Special relativity replaces the Newtonian ideas of absolute space and absolute time with spacetime. Its key ingredients are:

- the laws of physics are the same in all inertial frames,
- the speed of light in vacuum is the same for all inertial observers,
- time and space are not absolute separately, but are linked in spacetime.

A central consequence is that different observers can disagree on time intervals and spatial distances, but agree on the spacetime interval.

7.3 Minkowski spacetime and the interval

In flat spacetime, the invariant interval is

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2.$$

In spherical coordinates this becomes

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

Classification of intervals:

- $ds^2 < 0$: timelike
- $ds^2 = 0$: null
- $ds^2 > 0$: spacelike

For light,

$$ds = 0.$$

Thus for radial light propagation in flat spacetime,

$$0 = -c^2dt^2 + dr^2 \quad \Rightarrow \quad \frac{dr}{dt} = \pm c.$$

7.4 Proper time and geodesics

The **proper time** is the time measured by a clock moving with the object. In special relativity, inertial objects move on straight worldlines in spacetime. In general relativity, freely falling objects move on geodesics of curved spacetime.

This is the relativistic replacement for the Newtonian idea that gravity is a force. Instead, freely falling objects follow the natural geometry of spacetime.

7.5 Curved geometry intuition

A major geometric lesson for cosmology is that curved spaces do not obey all Euclidean rules.

Examples:

- the circumference of a circle need not be $2\pi r$,
- the angles of a triangle need not add to π,
- the angular size of an object need not decrease monotonically with distance.

For a 2D sphere of radius R, a circle of geodesic radius r has circumference

$$C = 2\pi R \sin(r/R).$$

This is smaller than $2\pi r$ for $r > 0$, which is the signature of positive curvature.

7.6 Robertson–Walker metric

If the universe is homogeneous and isotropic, the spacetime metric must take the Robertson–Walker form:

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_k(r)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right].$$

Here:
- t is cosmic time,
- $a(t)$ is the scale factor,
- r, θ, ϕ are comoving spatial coordinates,
- k labels the spatial curvature.

The function $S_k(r)$ depends on curvature:

$$S_k(r) = \begin{cases} R_0 \sin(r/R_0), & k = +1 \\ r, & k = 0 \\ R_0 \sinh(r/R_0), & k = -1 \end{cases}$$

Thus:
- $k = +1$: positively curved / closed
- $k = 0$: flat
- $k = -1$: negatively curved / open

7.7 Proper distance and comoving coordinates

In an expanding universe, galaxies that participate in the Hubble flow remain at fixed comoving coordinate r. Their **proper distance** changes because the scale factor changes:

$$d_P(t) = a(t)\, r.$$

Thus:
- comoving coordinates remain fixed for the background expansion,
- proper distances grow as the universe expands.

This is the geometric basis of Hubble expansion.

7.8 Physical meaning of the scale factor

The scale factor $a(t)$ measures the relative size of the universe. If

$$a(t_2) > a(t_1),$$

then all large-scale proper distances are larger at t_2 than at t_1.

The standard convention is

$$a_0 = 1$$

today.

Then

$$1 + z = \frac{1}{a_{\text{em}}}.$$

7.9 Large-scale electric neutrality

The universe must be electrically neutral to extremely high precision on large scales. The reason is that the electromagnetic force is vastly stronger than gravity. If there were even a small net cosmic charge imbalance, electromagnetic forces would dominate cosmic dynamics, contrary to observation.

Thus the fact that gravity governs large-scale structure strongly supports large-scale electric neutrality.

7.10 High-yield Chapter 3 formulas

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

$$ds^2 = -c^2dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$ds = 0 \quad \text{for light}$$

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_k(r)^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$d_P(t) = a(t)\, r$$

For a positively curved 2D sphere:

$$C = 2\pi R \sin(r/R)$$

7.11 Typical Chapter 3 exam asks

- Explain the difference between Newtonian and relativistic gravity.
- Use $ds = 0$ to derive the speed/path of light in flat spacetime.
- Interpret the Robertson–Walker metric physically.
- Identify what $a(t)$, r, and k mean.
- Solve a curvature/geometry problem using spherical geometry.
- Explain why the universe must be electrically neutral on large scales.

7.12 Chapter 3 traps / reminders

- Do not confuse coordinate distance with proper distance.
- Do not forget that $a_0 = 1$ is a convention.
- Do not treat k as an energy-density component; it is a geometry label.
- $k = +1, 0, -1$ tells the sign of spatial curvature, not the instantaneous expansion rate.
- In curved geometry, Euclidean intuition can fail badly.

8 Chapter 4 — Cosmic Dynamics

8.1 From Einstein's equation to cosmology

General relativity relates spacetime curvature to energy and momentum through Einstein's field equation. In cosmology, once homogeneity and isotropy are imposed, the full tensor equation reduces to a small set of scalar equations governing the expansion of the universe.

The three central equations are:
- the Friedmann equation,
- the fluid (continuity) equation,
- the acceleration equation.

Together they describe how the scale factor evolves in response to the energy content of the universe.

8.2 The Friedmann equation

The Friedmann equation is

$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3},$$

where

$$H = \frac{\dot{a}}{a}.$$

Interpretation of terms:

- $\frac{8\pi G}{3c^2}\varepsilon$: expansion driven by total energy density,
- $-\frac{kc^2}{R_0^2 a^2}$: contribution of spatial curvature,
- $\frac{\Lambda c^2}{3}$: cosmological constant / vacuum-energy contribution.

The Friedmann equation is the fundamental cosmic expansion-rate equation.

8.3 Critical density

The **critical density** is the density required for a spatially flat universe ($k = 0$) if Λ is handled separately:

$$\varepsilon_c = \frac{3c^2 H^2}{8\pi G}.$$

At the present time,

$$\varepsilon_{c,0} = \frac{3c^2 H_0^2}{8\pi G}.$$

Density parameters are defined by

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_c}.$$

For the total density parameter,

$$\Omega - 1 = \frac{kc^2}{H^2 R_0^2 a^2}.$$

Thus:

$$k = +1 \Rightarrow \Omega > 1, \qquad k = 0 \Rightarrow \Omega = 1, \qquad k = -1 \Rightarrow \Omega < 1.$$

8.4 The fluid / continuity equation

The continuity equation is

$$\dot{\varepsilon} = -3H(\varepsilon + P).$$

This expresses energy conservation in an expanding universe. It says that the energy density changes because:

- the volume changes as the universe expands,
- pressure can do work.

This equation is one of the most useful in the course, because it directly gives the scaling of density with the scale factor.

8.5 The acceleration equation

The acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda c^2}{3}.$$

This equation tells whether the expansion speeds up or slows down.

Key lesson:

$$\varepsilon + 3P$$

determines the decelerating effect of matter/radiation, while Λ contributes acceleration.

8.6 Pressure as a source of gravity

A major conceptual difference between Newtonian gravity and GR is that **pressure gravitates**. In cosmology, it is not enough to know the energy density alone.

Examples:

- matter has $P \approx 0$,
- radiation has $P = \varepsilon/3$,
- vacuum energy has $P = -\varepsilon$.

Thus radiation decelerates the expansion more strongly than matter at the same energy density, while vacuum energy can accelerate the expansion.

8.7 Equation-of-state parameter w

A very useful parameterization is

$$P = w\varepsilon.$$

Important cases:

$$w = 0 \qquad \text{matter}$$

$$w = \frac{1}{3} \qquad \text{radiation}$$

$$w = -1 \qquad \text{cosmological constant}$$

Acceleration occurs when

$$\varepsilon + 3P < 0 \quad \Rightarrow \quad w < -\frac{1}{3}.$$

Thus a component with sufficiently negative pressure drives accelerated expansion.

8.8 Scaling with the scale factor

Insert $P = w\varepsilon$ into the continuity equation:

$$\dot{\varepsilon} = -3H\varepsilon(1+w).$$

Using $H = \dot{a}/a$,

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3(1+w)\frac{\dot{a}}{a}.$$

Integrating,

$$\varepsilon \propto a^{-3(1+w)}.$$

Important special cases:

$$\varepsilon_m \propto a^{-3}$$

$$\varepsilon_r \propto a^{-4}$$

$$\varepsilon_\Lambda = \text{constant}$$

The extra factor of a^{-1} for radiation appears because photon energies redshift as the universe expands.

8.9 Curvature as an effective component

Although curvature is geometric rather than material, its Friedmann contribution scales as

$$a^{-2}.$$

Thus the different important scalings are:

$$\varepsilon_r \propto a^{-4}, \qquad \varepsilon_m \propto a^{-3}, \qquad \varepsilon_k \propto a^{-2}, \qquad \varepsilon_\Lambda \propto a^{0}.$$

This ordering is crucial for understanding cosmic eras.

8.10 Einstein static universe

Einstein originally tried to construct a static universe by balancing the attractive gravity of matter against the repulsive effect of a cosmological constant.

This balance is unstable. A small perturbation causes the universe either to expand or contract.

A useful conceptual result from this chapter is:

- converting matter into radiation increases the gravitating effect because radiation has pressure,
- therefore an Einstein static universe would contract if some matter were converted into radiation while Λ stayed fixed.

8.11 Cosmological constant as vacuum energy

A cosmological constant behaves like a fluid with

$$P_\Lambda = -\varepsilon_\Lambda.$$

So

$$w_\Lambda = -1.$$

Because its pressure is negative enough,

$$\varepsilon_\Lambda + 3P_\Lambda = \varepsilon_\Lambda - 3\varepsilon_\Lambda = -2\varepsilon_\Lambda < 0,$$

it drives accelerated expansion.

8.12 How to think about the three main equations

A useful summary:

- Friedmann equation: gives the *expansion rate*

$$H = \frac{\dot{a}}{a}$$

  from the contents of the universe.
- Continuity equation: tells how *densities evolve* as the universe expands.
- Acceleration equation: tells whether the expansion *speeds up or slows down*.

8.13 High-yield Chapter 4 formulas

$$H = \frac{\dot{a}}{a}$$

$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

$$\varepsilon_c = \frac{3c^2H^2}{8\pi G}$$

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_c}$$

$$\dot{\varepsilon} = -3H(\varepsilon + P)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda c^2}{3}$$

$$P = w\varepsilon$$

$$\varepsilon \propto a^{-3(1+w)}$$

8.14 Typical Chapter 4 exam asks

- State and interpret the Friedmann equation.
- Derive or use the density scaling law from the continuity equation.
- Determine whether a component accelerates or decelerates expansion.
- Relate the sign of curvature to Ω.
- Explain the role of pressure in GR.
- Explain why radiation gravitates more strongly than matter at the same energy density.
- Analyze Einstein's static universe conceptually.

8.15 Chapter 4 traps / reminders

- Do not forget the factor of $3P$ in the acceleration equation.
- Do not confuse total Ω with individual Ω_i.
- The curvature term is not matter; it is geometric.
- A positive Λ contributes acceleration, not deceleration.
- Radiation has $w = \frac{1}{3}$, not $w = 1$.
- Acceleration requires $w < -\frac{1}{3}$, not merely $w < 0$.

9 Chapter 5 — Model Universes

9.1 Why model universes matter

Once the Friedmann, continuity, and acceleration equations are known, the next step is to study simple universes with one or a few dominant components. These models are useful because:

- they give intuition for how different components affect expansion,
- many exam problems reduce to single-component or two-component limits,
- they provide exact or parametric solutions for $a(t)$, $H(t)$, and cosmic age.

9.2 One-component flat universes

For a flat universe with a single component of constant equation-of-state parameter w,

$$\varepsilon \propto a^{-3(1+w)}.$$

Using the Friedmann equation,

$$H^2 \propto \varepsilon \propto a^{-3(1+w)},$$

so

$$H(a) = H_0\, a^{-3(1+w)/2}.$$

Since

$$H = \frac{\dot{a}}{a},$$

the scale factor evolves as

$$a(t) \propto t^{2/[3(1+w)]} \qquad (w > -1).$$

This is one of the most important formulas in the course.

9.3 Matter-only universe

For pressureless matter,

$$w = 0, \qquad \varepsilon_m \propto a^{-3}.$$

Thus

$$a(t) \propto t^{2/3}.$$

This universe expands, but decelerates. The age of a flat matter-only universe is

$$t_0 = \frac{2}{3H_0}.$$

9.4 Radiation-only universe

For radiation,

$$w = \frac{1}{3}, \qquad \varepsilon_r \propto a^{-4}.$$

Thus

$$a(t) \propto t^{1/2}.$$

Radiation-dominated expansion decelerates more strongly than matter-dominated expansion because radiation contributes pressure as well as energy density.

9.5 Lambda-only universe

For a cosmological constant,

$$w = -1, \qquad \varepsilon_\Lambda = \text{constant}.$$

Then H is constant, so

$$\frac{\dot{a}}{a} = H \quad \Rightarrow \quad a(t) \propto e^{Ht}.$$

Thus a Λ-dominated universe expands exponentially.

9.6 Empty universe

An empty universe has essentially no matter, no radiation, and no cosmological constant. In the simplest Milne-type case, the expansion is linear:

$$a(t) \propto t.$$

This model is useful as a limiting case and for building intuition, even though it is not a realistic description of our universe.

9.7 Curved matter-only universes

For matter-only universes with nonzero curvature, the scale factor is not a simple power law at all times. Instead, positively and negatively curved cases are described by parametric solutions.

Positively curved matter-only universe ($k = +1,\ \Omega_0 > 1$)

The standard parametric solution is

$$a(\theta) = \frac{\Omega_0}{2(\Omega_0 - 1)}(1 - \cos\theta),$$

$$t(\theta) = \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta).$$

This universe expands from a Big Bang, reaches a maximum size, then recollapses in a Big Crunch.

Negatively curved matter-only universe ($k = -1,\ \Omega_0 < 1$)

The standard parametric solution is

$$a(\eta) = \frac{\Omega_0}{2(1 - \Omega_0)}(\cosh\eta - 1),$$

$$t(\eta) = \frac{\Omega_0}{2H_0(1 - \Omega_0)^{3/2}}(\sinh\eta - \eta).$$

This universe expands forever and is older than the flat matter-only case for fixed H_0.

9.8 Age of the universe in matter-only models

The age depends strongly on curvature:

- flat matter-only:
$$t_0 = \frac{2}{3H_0}$$
- positively curved matter-only ($\Omega_0 > 1$): younger than the flat matter-only case
- negatively curved matter-only ($\Omega_0 < 1$): older than the flat matter-only case

Thus the present age contains information about the composition and geometry of the universe.

9.9 Redshift drift

A source observed at redshift z at time t_0 will have its redshift change slowly with the observer's time. In a flat, one-component universe with constant w,

$$\frac{dz}{dt_0} = H_0\left[(1+z) - (1+z)^{\frac{3}{2}(1+w)}\right].$$

This shows that redshift need not be literally constant in time. The sign of the drift depends on the cosmic expansion history.

In particular, the observed redshift increases with time when

$$w < -\frac{1}{3}.$$

9.10 Multi-component universes

Realistic universes contain multiple components:

- radiation,
- matter,
- curvature,
- Λ / dark energy.

Then

$$\frac{H(a)^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}.$$

As the universe expands, different components dominate at different times because they scale differently with a.

9.11 Equality epochs and domination transitions

The ordering of scalings is

$$a^{-4}, \qquad a^{-3}, \qquad a^{-2}, \qquad a^{0}.$$

Thus:

- at very early times, radiation dominates,
- later, matter dominates,
- at late times in the benchmark model, Λ dominates.

Radiation–matter equality occurs when

$$\varepsilon_r = \varepsilon_m,$$

so

$$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}}.$$

Matter–Λ equality occurs when

$$\varepsilon_m = \varepsilon_\Lambda.$$

These transitions are important for understanding the thermal and structural history of the universe.

9.12 Benchmark model

A useful modern benchmark model is spatially flat with

$$\Omega_{r,0} \approx 9 \times 10^{-5}, \qquad \Omega_{m,0} \approx 0.31, \qquad \Omega_{\Lambda,0} \approx 0.69.$$

Its qualitative expansion history is:

- radiation-dominated early universe,
- matter-dominated intermediate era,
- Λ-dominated late-time accelerated era.

9.13 Phantom energy and the Big Rip

If a component has

$$w < -1,$$

its energy density grows with expansion:

$$\varepsilon \propto a^{-3(1+w)}.$$

This leads to runaway accelerated expansion, and the scale factor can diverge at a finite future time. This is called the **Big Rip**.

In a flat universe dominated by phantom energy,

$$t_{\text{rip}} - t_0 = \frac{2}{3H_0\sqrt{\Omega_{p,0}}\,|1+w_p|}.$$

9.14 How to think about Chapter 5 models

A useful logic chain is:

$$w \to \varepsilon(a) \to H(a) \to a(t) \to \text{age / fate}$$

If you know w, you can usually reconstruct the rest.

9.15 High-yield Chapter 5 formulas

$$\varepsilon \propto a^{-3(1+w)}$$

$$H(a) = H_0\, a^{-3(1+w)/2} \qquad \text{(flat one-component universe)}$$

$$a(t) \propto t^{2/[3(1+w)]}$$

$$a(t) \propto t^{2/3} \qquad \text{matter-only}$$

$$a(t) \propto t^{1/2} \qquad \text{radiation-only}$$

$$a(t) \propto e^{Ht} \qquad \Lambda\text{-only}$$

$$\frac{H(a)^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}$$

$$\frac{dz}{dt_0} = H_0\left[(1+z) - (1+z)^{\frac{3}{2}(1+w)}\right]$$

$$a_{rm} = \frac{\Omega_{r,0}}{\Omega_{m,0}}$$

9.16 Typical Chapter 5 exam asks

- Given w, derive $\varepsilon(a)$, $H(a)$, or $a(t)$.
- Compare matter-, radiation-, and Λ-dominated universes.
- Compute age in curved matter-only universes.
- Determine which component dominates at a given epoch.
- Interpret redshift drift.
- Explain the qualitative fate of a phantom-energy universe.

9.17 Chapter 5 traps / reminders

- Do not confuse a one-component flat result with a multicomponent realistic universe.
- Do not forget that curvature behaves as a^{-2}, not a^{-3}.
- Radiation domination and matter domination have different $a(t)$ laws.
- A Λ-dominated universe has accelerated expansion.
- Phantom energy has $w < -1$, not merely $w < -\frac{1}{3}$.

10 Chapter 6 — Cosmological Distances

10.1 Why cosmological distances are subtle

In an expanding universe, distance is not a unique concept. Different distance definitions are useful for different observables. The three most important are:

- proper distance,
- luminosity distance,
- angular-diameter distance.

Many exam problems in cosmology reduce to identifying which of these is relevant.

10.2 Proper distance

The proper distance to an object at comoving coordinate r at cosmic time t is

$$d_P(t) = a(t)\, r.$$

At the present time,

$$d_P(t_0) = a_0 r = r$$

if $a_0 = 1$.

More generally, proper distance today can be written as

$$d_P(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}.$$

This is the direct distance along a constant-time slice of the Robertson–Walker geometry.

10.3 Standard candles and luminosity

A **standard candle** is an object whose luminosity L is known or can be inferred. The observed flux f is related to luminosity distance d_L by

$$f = \frac{L}{4\pi d_L^2}.$$

Thus,

$$d_L = \sqrt{\frac{L}{4\pi f}}.$$

This is the distance measure relevant for supernovae and magnitude problems.

10.4 Apparent and absolute magnitude

The apparent magnitude m measures how bright an object appears. The absolute magnitude M is the apparent magnitude the object would have if placed at a standard distance of 10 pc.

They are related by the distance modulus:

$$m - M = 5\log_{10}\left(\frac{d_L}{10\text{ pc}}\right).$$

Important conceptual point: smaller magnitude means brighter source.

10.5 Luminosity distance

In cosmology,

$$d_L = (1+z)d_P(t_0).$$

The extra factor of $1+z$ relative to proper distance arises because:

- each photon is redshifted in energy,
- the photon arrival rate is time-dilated,
- spatial geometry also contributes.

Thus luminosity distance grows faster with redshift than proper distance.

10.6 Angular-diameter distance

A **standard yardstick** is an object of known physical size ℓ. If its observed angular size is $\delta\theta$, then

$$d_A = \frac{\ell}{\delta\theta}$$

for small angles.

Angular-diameter distance is related to proper distance by

$$d_A = \frac{d_P(t_0)}{1+z},$$

and hence

$$d_A = \frac{d_L}{(1+z)^2}.$$

This is one of the most important relations in the chapter.

10.7 Why angular size can reach a minimum

In Euclidean intuition, more distant objects always look smaller. In cosmology, that is not always true. Because of expansion and spacetime geometry, d_A can first increase with redshift, then decrease.

Thus very distant objects can appear *larger* again in angular size. Equivalently:

$$d_A \text{ can have a maximum as a function of } z.$$

This is a favorite conceptual exam point.

10.8 Distance relations summary

The main relations are:

$$d_P(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)}$$

$$d_L = (1+z)d_P$$

$$d_A = \frac{d_P}{1+z}$$

$$d_A = \frac{d_L}{(1+z)^2}$$

These should be memorized.

10.9 Flat one-component universes: distance formulas

For a spatially flat one-component universe with constant w, the proper distance today is

$$d_P(t_0) = \frac{2c}{H_0(1+3w)}\left[1-(1+z)^{-(1+3w)/2}\right] \qquad (w \neq -\tfrac{1}{3}).$$

Then

$$d_L = (1+z)d_P$$

and

$$d_A = \frac{d_P}{1+z}.$$

Special case $w = -\frac{1}{3}$:

$$d_P(t_0) = \frac{c}{H_0}\ln(1+z),$$

$$d_L = \frac{c}{H_0}(1+z)\ln(1+z),$$

$$d_A = \frac{c}{H_0}\frac{\ln(1+z)}{1+z}.$$

These formulas are useful for explicit derivation problems.

10.10 Maximum angular-diameter distance in one-component universes

For $w \neq -\frac{1}{3}, -1$, the maximum angular-diameter distance occurs at

$$z_c = \left[\frac{3(1+w)}{2}\right]^{2/(1+3w)} - 1.$$

The maximum value is

$$d_{A,\max} = \frac{c}{H_0}\left[\frac{2}{3(1+w)}\right]^{\frac{3(1+w)}{1+3w}}.$$

Special cases:

$$w = 0 \Rightarrow z_c = \frac{5}{4}, \qquad d_{A,\max} = \frac{8}{27}\frac{c}{H_0}$$

$$w = \frac{1}{3} \Rightarrow z_c = 1, \qquad d_{A,\max} = \frac{1}{4}\frac{c}{H_0}$$

$$w = -\frac{1}{3} \Rightarrow z_c = e-1, \qquad d_{A,\max} = \frac{1}{e}\frac{c}{H_0}$$

10.11 Physical interpretation of the three distances

- d_P: geometric distance on a constant-time slice.
- d_L: the distance that makes inverse-square flux law look Euclidean.
- d_A: the distance that makes angular size look Euclidean.

The same object generally has different values of these distances at the same redshift.

10.12 Why this chapter matters observationally

Chapter 6 connects cosmological theory to observations:

- supernova cosmology uses d_L,
- standard rulers use d_A,
- the spacetime geometry determines how these differ from simple Euclidean expectations.

10.13 High-yield Chapter 6 formulas

$$f = \frac{L}{4\pi d_L^2}$$

$$m - M = 5\log_{10}\left(\frac{d_L}{10\text{ pc}}\right)$$

$$d_P(t_0) = c\int_{t_e}^{t_0}\frac{dt}{a(t)}$$

$$d_L = (1+z)d_P$$

$$d_A = \frac{d_P}{1+z}$$

$$d_A = \frac{d_L}{(1+z)^2}$$

$$\delta\theta = \frac{\ell}{d_A}$$

For flat one-component universes:

$$d_P(t_0) = \frac{2c}{H_0(1+3w)}\left[1-(1+z)^{-(1+3w)/2}\right] \qquad (w \neq -\tfrac{1}{3})$$

10.14 Typical Chapter 6 exam asks

- Use L and f to find d_L.
- Use m, M, and the distance modulus.
- Given ℓ and $\delta\theta$, find d_A.
- Relate d_L, d_A, and d_P.
- Find or interpret the redshift where angular size is minimal.
- Derive d_P, d_L, or d_A in a one-component universe.

10.15 Chapter 6 traps / reminders

- Do not confuse luminosity distance with angular-diameter distance.
- Do not forget the factors of $1+z$:

$$d_L = (1+z)d_P, \qquad d_A = \frac{d_P}{1+z}.$$

- The same object can have different cosmological distance measures.
- Apparent magnitude decreases as brightness increases.
- Angular size need not decrease monotonically with redshift.

11 Chapter 7 — Dark Matter

11.1 Why dark matter is needed

The observed luminous matter in the universe is not sufficient to explain the gravitational behavior of galaxies and clusters. Multiple independent observations show that there is substantially more gravitating matter than can be accounted for by stars, gas, and dust alone.

The main evidence comes from:

- galaxy rotation curves,
- virial masses of clusters and dwarf galaxies,
- gravitational lensing,
- the broader cosmological matter budget.

11.2 Visible matter vs dark matter

Visible matter includes stars, luminous gas, dust, and other matter detected electromagnetically.

Dark matter is matter inferred gravitationally but not directly observed through ordinary electromagnetic emission or absorption. It must:

- contribute mass,
- cluster gravitationally,
- not emit significant light,
- not be made primarily of normal visible stars.

11.3 Galaxy rotation curves

Consider a star orbiting at radius R around a galaxy with orbital speed $v(R)$. Newtonian circular motion gives

$$\frac{v^2}{R} = \frac{GM(R)}{R^2},$$

so

$$M(R) = \frac{v^2 R}{G}.$$

If most of a galaxy's mass were concentrated where the light is, then beyond the luminous disk one would expect

$$v(R) \propto R^{-1/2}.$$

Instead, observed rotation curves are often approximately flat:

$$v(R) \approx \text{constant}.$$

Then

$$M(R) \propto R.$$

This implies that mass continues to grow with radius even where the light does not, which is evidence for extended dark halos.

11.4 Halo density profile from flat rotation curves

If v is approximately constant with radius, then

$$M(R) = \frac{v^2 R}{G}.$$

Differentiating,

$$\frac{dM}{dR} = \frac{v^2}{G}.$$

Since

$$\frac{dM}{dR} = 4\pi R^2 \rho(R),$$

we obtain

$$\rho(R) = \frac{v^2}{4\pi G R^2}.$$

Thus a simple halo giving a flat rotation curve has density scaling approximately as

$$\rho(R) \propto R^{-2}.$$

11.5 Local halo density and compact dark objects

If dark matter were made of compact objects of mass m, then their number density near the Sun would be roughly

$$n \sim \frac{\rho_{\rm halo}}{m}.$$

The typical nearest-neighbor distance would scale as

$$d_{\rm near} \sim n^{-1/3}.$$

The rate at which such objects pass within impact parameter b of the Sun is roughly

$$\Gamma \sim n v \pi b^2.$$

This kind of estimate is useful for MACHO or compact-object constraints.

11.6 Virial theorem in galaxies and clusters

A gravitationally bound system in equilibrium satisfies the virial theorem:

$$2K + U = 0.$$

This leads to order-of-magnitude mass estimates from observed velocity dispersions.

A useful form is

$$M \sim \frac{\langle v^2 \rangle r_h}{\alpha G},$$

where:

- $\langle v^2 \rangle$ is a characteristic velocity-squared,
- r_h is a characteristic radius such as the half-light radius,
- α is a dimensionless structure factor of order unity.

If only the line-of-sight dispersion σ_r is observed and the system is isotropic, then

$$\langle v^2 \rangle = 3\sigma_r^2.$$

Large mass-to-light ratios inferred this way are strong evidence for dark matter.

11.7 Mass-to-light ratio

The mass-to-light ratio is

$$\frac{M}{L}.$$

A system with a much larger M/L than expected from ordinary stellar populations is dark-matter dominated.

This is especially striking in dwarf spheroidal galaxies, which can have very large mass-to-light ratios even though they are faint.

11.8 MACHOs and compact dark matter

MACHOs (Massive Astrophysical Compact Halo Objects) were proposed as a possible dark-matter candidate category. Examples include:

- faint stars,
- brown dwarfs,
- stellar remnants,
- primordial black holes.

If the halo consisted of such compact objects, one could estimate:

- their local number density,
- nearest-object distance,
- close-passage frequency,
- microlensing signatures.

Although compact objects may contribute some mass, they do not appear to account for all dark matter.

11.9 Gravitational lensing basics

According to general relativity, light is deflected by mass. For a lens of mass M and impact parameter b, the weak-field deflection angle is

$$\alpha = \frac{4GM}{bc^2}.$$

For a grazing ray at the surface of an object of radius R,

$$\alpha = \frac{4GM}{Rc^2}.$$

Lensing allows mass to be measured independent of luminosity and is therefore powerful evidence for dark matter.

11.10 Why lensing is important

Lensing is especially valuable because:

- it measures total gravitating mass,
- it does not rely on dynamical equilibrium assumptions,
- it can map mass distributions directly.

Thus lensing provides an independent check on the need for dark matter.

11.11 Multiple lines of evidence

Dark matter is not inferred from one effect alone. The case is strong because different methods point to the same conclusion:

- rotation curves indicate extended halo mass,
- virial analyses indicate unseen cluster/galaxy mass,
- lensing shows more mass than visible matter provides,
- cosmology indicates baryons alone are insufficient.

11.12 High-yield Chapter 7 formulas

$$M(R) = \frac{v^2 R}{G}$$

$$\rho(R) = \frac{v^2}{4\pi G R^2} \qquad \text{(flat rotation curve halo)}$$

$$M \sim \frac{\langle v^2 \rangle r_h}{\alpha G} \qquad \text{(virial estimate)}$$

$$\langle v^2 \rangle = 3\sigma_r^2 \qquad \text{(isotropic system)}$$

$$\frac{M}{L}$$

$$\alpha = \frac{4GM}{bc^2} \qquad \text{(lensing deflection)}$$

11.13 Typical Chapter 7 exam asks

- Use a flat rotation curve to infer $M(R)$ or $\rho(R)$.
- Use a velocity dispersion and radius to estimate virial mass.
- Compute a mass-to-light ratio and interpret it.
- Estimate nearest-neighbor distances or encounter rates for compact halo objects.
- Compute the deflection of light grazing a massive body.
- Explain why dark matter is required observationally.

11.14 Chapter 7 traps / reminders

- Do not assume that light traces mass.
- Flat rotation curves imply $M(R) \propto R$, not constant enclosed mass.
- Virial estimates assume approximate equilibrium; this is a possible source of uncertainty.
- Lensing depends on total mass, not just visible mass.
- A large M/L does not by itself tell you what dark matter is, only that unseen mass is present.

12 Chapter 8 — CMB and Recombination

12.1 Why the CMB matters

The cosmic microwave background gives a snapshot of the universe near the time of last scattering. It is one of the strongest pieces of evidence for the Hot Big Bang model because:

- it fills all space,
- it has an almost perfect blackbody spectrum,
- it preserves information about the early universe.

The CMB tells us about the state of the universe when photons last scattered efficiently off matter.

12.2 Recombination

Recombination is the epoch when free electrons and protons combined to form neutral hydrogen:

$$p + e^- \leftrightarrow H + \gamma.$$

This process reduces the free-electron density and therefore greatly reduces the Thomson scattering rate of photons.

A key benchmark value is

$$T_{\rm rec} \approx 3760 \text{ K}.$$

12.3 Decoupling and last scattering

Recombination and **decoupling** are related but not identical:

- recombination refers to the formation of neutral atoms,
- decoupling refers to photons ceasing to scatter efficiently and beginning to free-stream.

The photons we see in the CMB come from the **last scattering surface**. A useful benchmark is

$$z_{\rm dec} \approx 1090.$$

12.4 Why recombination is delayed

A naive thought is that hydrogen should recombine when

$$kT \sim 13.6 \text{ eV}.$$

In reality, recombination occurs at a much lower temperature. The reason is that the photon-to-baryon ratio is huge:

$$\eta^{-1} \sim 10^9.$$

Even when the average photon energy is far below 13.6 eV, the high-energy tail of the blackbody spectrum still contains enough ionizing photons to keep hydrogen ionized.

Thus recombination is delayed until the high-energy tail becomes sufficiently depleted.

12.5 The Saha equation

When photoionization and recombination are in chemical equilibrium, the ionization fraction can be found using the Saha equation.

Define the ionization fraction

$$X \equiv \frac{n_p}{n_{\rm bary}} = \frac{n_e}{n_{\rm bary}}.$$

Then the hydrogen recombination problem can be written in the form

$$\frac{X^2}{1-X} = S(T,\eta),$$

where $S(T,\eta)$ is a known function of temperature and baryon-to-photon ratio.

Solving the quadratic gives

$$X(T,\eta) = \frac{-S + \sqrt{S^2+4S}}{2}.$$

This form is extremely useful for exam problems.

12.6 Dependence of recombination on η

The baryon-to-photon ratio η affects recombination through the Saha equation. Larger η means more baryons per photon, which shifts recombination to a slightly higher temperature.

However, the dependence is weak because the ionization fraction depends exponentially on Q/kT. Thus $T_{\rm rec}$ changes only modestly when η changes by order unity.

12.7 Defining the recombination temperature

A practical definition is to take the recombination temperature $T_{\rm rec}$ as the temperature where

$$X = \frac{1}{2}.$$

Then

$$\frac{X^2}{1-X} = \frac{1}{2},$$

so

$$S(T_{\rm rec},\eta) = \frac{1}{2}.$$

This is the condition used in many Chapter 8 problems.

12.8 One ionizing photon per baryon

A related and important concept is the temperature at which there is about one ionizing photon per baryon. This depends on the high-energy tail of the blackbody distribution, not just on the average photon energy.

A useful notation is

$$F(>E_0) = F(E > E_0),$$

the fraction of blackbody photons with energy greater than E_0.

For $E_0 \gg kT$,

$$F(>E_0) \approx \frac{(x^2+2x+2)e^{-x}}{2\zeta(3)}, \qquad x = \frac{E_0}{kT}.$$

Here $2\zeta(3)$ comes from the normalization of the total blackbody photon number density.

12.9 Benchmark values

Important values to remember:

$$T_0 = 2.7255 \text{ K}$$

$$n_\gamma \approx 411 \text{ cm}^{-3}$$

$$\eta \approx 6 \times 10^{-10}$$

$$T_{\text{rec}} \approx 3760 \text{ K}$$

$$z_{\text{rec}} \approx 1380$$

$$z_{\text{dec}} \approx 1090$$

12.10 Helium recombination / variant Saha problems

The same equilibrium logic used for hydrogen can be adapted to other atomic species such as helium. In such problems:

- replace the ionization energy Q,
- replace the statistical-weight factor,
- carefully define the relevant number density.

This is useful for generalized recombination exercises.

12.11 What the CMB is actually showing us

The CMB is not a picture of the Big Bang itself. It is a picture of the universe at the epoch when photons last scattered efficiently. Because the earlier universe was opaque, direct electromagnetic observations cannot probe much earlier than this.

Thus the CMB is a *surface of last scattering*, not a direct image of $t = 0$.

12.12 Why Chapter 8 matters

Chapter 8 connects thermodynamics, atomic physics, and cosmology. It explains:

- why the universe became transparent,
- why recombination happens when it does,
- why the CMB exists and carries early-universe information.

12.13 High-yield Chapter 8 formulas

$$p + e^- \leftrightarrow H + \gamma$$

$$X \equiv \frac{n_p}{n_{\text{bary}}}$$

$$\frac{X^2}{1-X} = S(T,\eta)$$

$$X(T,\eta) = \frac{-S + \sqrt{S^2+4S}}{2}$$

For $X = \frac{1}{2}$,

$$S(T_{\text{rec}},\eta) = \frac{1}{2}$$

For the blackbody high-energy tail,

$$F(>E_0) \approx \frac{(x^2+2x+2)e^{-x}}{2\zeta(3)}, \qquad x = \frac{E_0}{kT}$$

12.14 Typical Chapter 8 exam asks

- Explain the difference between recombination and decoupling.
- Use the Saha equation to solve for $X(T,\eta)$ or T_{rec}.
- Explain why recombination occurs well below 13.6 eV.
- Use the blackbody tail fraction to estimate when one ionizing photon per baryon remains.
- Interpret benchmark values $T_{\text{rec}}, z_{\text{rec}}, z_{\text{dec}}$.

12.15 Chapter 8 traps / reminders

- Do not confuse recombination with decoupling.
- The average photon energy being below 13.6 eV is not enough for recombination.
- Keep track of whether X is defined as ionized fraction or neutral fraction.
- $F(>E_0)$ means the fraction of photons with energy greater than E_0.
- The denominator $2\zeta(3)$ in the tail-fraction formula comes from the total blackbody photon number normalization.

13 Chapter 9 — Nucleosynthesis and the Early Universe

13.1 Why Big Bang nucleosynthesis matters

The cosmic microwave background gives direct information about the universe at last scattering, but the universe was opaque before that time. To probe earlier epochs, we rely on theoretical physics combined with present-day relic abundances.

Big Bang nucleosynthesis (BBN) explains the primordial production of the light nuclei:

- ^{1}H,
- ^{2}H (deuterium),
- ^{3}He,
- ^{4}He,
- trace ^{7}Li.

It occurs when the universe is only a few minutes old, at temperatures of order

$$T_{\text{nuc}} \sim 10^9 \text{ K}.$$

Ryden, Chapter 9, pp. 1–5. [oai$_c$*itation* : 2*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.2 Nuclear binding and the light nuclei

Nuclear reactions are governed by nuclear binding energies, which are of order MeV rather than eV. In particular, deuterium has binding energy

$$B_D = 2.22 \text{ MeV},$$

and ^{4}He is especially tightly bound among light nuclei, with

$$B_{^4\text{He}} = 28.30 \text{ MeV}.$$

This explains why BBN, once it gets started, tends to drive material efficiently toward ^{4}He. *Ryden, Chapter 9, pp. 2–4.* [oai$_c$*itation* : 3*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.3 Neutron–proton equilibrium

At high temperature, neutrons and protons are interconverted by weak interactions and remain in thermal equilibrium. Their equilibrium number-density ratio is

$$\frac{n_n}{n_p} \approx e^{-Q_n/kT},$$

where

$$Q_n = (m_n - m_p)c^2 = 1.29 \text{ MeV}.$$

At very high temperature, $kT \gg Q_n$, so neutrons and protons are nearly equally abundant. As the temperature drops below the MeV scale, protons are increasingly favored and the neutron-to-proton ratio decreases exponentially. *Ryden, Chapter 9, pp. 6–9.*

13.4 Freezeout of the neutron–proton ratio

The weak interactions that maintain neutron–proton equilibrium become too slow compared with the expansion rate as the universe cools. Freezeout occurs when the weak interaction rate Γ falls to the Hubble rate H.

A key benchmark result is

$$kT_{\text{freeze}} \approx 0.8 \text{ MeV}, \qquad T_{\text{freeze}} \approx 9 \times 10^9 \text{ K}, \qquad t_{\text{freeze}} \sim 1 \text{ s}.$$

At freezeout,

$$\left(\frac{n_n}{n_p}\right)_{\text{freeze}} = e^{-Q_n/kT_{\text{freeze}}} \approx \frac{1}{5}.$$

Thus, at times just after freezeout, there is about one neutron for every five protons. *Ryden, Chapter 9, pp. 8–10.* [oai$_c$*itation* : 5*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.5 Neutron decay

A free neutron is unstable:

$$n \to p + e^- + \bar{\nu}_e,$$

with decay time

$$\tau_n = 880 \text{ s}.$$

Thus if nucleosynthesis does not begin immediately after freezeout, some neutrons decay before they can be incorporated into nuclei. If the time delay is t, the surviving neutron fraction is

$$f = e^{-t/\tau_n}.$$

This reduces the neutron-to-proton ratio and therefore lowers the maximum possible helium fraction. *Ryden, Chapter 9, pp. 5–6, 15–16.* [oai$_c$*itation* : 6*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.6 Why proton–proton fusion is inefficient in the early universe

A neutron can fuse readily with a proton to form deuterium:

$$p + n \to D + \gamma.$$

By contrast, proton–proton fusion is much slower because it must proceed through weak interactions. This is why BBN uses the neutrons already present after freezeout rather than converting large numbers of extra protons into neutrons or deuterons.

Thus the scarcity of neutrons is the main reason BBN is incomplete and leaves most baryons as ordinary hydrogen. *Ryden, Chapter 9, pp. 9–11.* [oai$_c$*itation* : 7*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.7 Maximum helium fraction

If every neutron available at the start of nucleosynthesis were incorporated into ^{4}He, then the maximum possible helium mass fraction would be

$$Y_{\max} = \frac{2f_n}{1+f_n}, \qquad f_n \equiv \frac{n_n}{n_p}.$$

For $f_n = 1/5$,

$$Y_{\max} = \frac{2(1/5)}{1+1/5} = \frac{1/3}{1} = 0.33.$$

Because some neutrons decay before BBN begins, the actual maximum drops to about

$$Y_{\max} \approx 0.27.$$

The observed primordial helium fraction is

$$Y_p \approx 0.24,$$

so the observed value is slightly below the theoretical maximum, as expected. *Ryden, Chapter 9, pp. 11–16.* [oai$_c$*itation* : 8*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.8 Deuterium synthesis and the deuterium bottleneck

The first essential step of BBN is deuterium formation:

$$p + n \leftrightarrow D + \gamma.$$

This reaction is analogous to recombination in Chapter 8: two particles bind together and a photon carries away the excess energy.

Although the binding energy of deuterium is 2.22 MeV, deuterium does not become abundant as soon as $kT \ll 2.22$ MeV. The reason is that the enormous photon-to-baryon ratio means there are still enough high-energy photons in the blackbody tail to photodissociate deuterium. This delay is the **deuterium bottleneck**. *Ryden, Chapter 9, pp. 12–15.* [oai$_c$*itation* : 9*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.9 Saha-like equation for deuterium

The deuteron-to-neutron ratio is governed by a nucleosynthetic analogue of the Saha equation. The chapter gives a simplified expression for n_D/n_n as a function of T and η, and defines the nucleosynthesis temperature T_{nuc} by the condition

$$\frac{n_D}{n_n} = 1.$$

Using the benchmark baryon-to-photon ratio

$$\eta \approx 6.1 \times 10^{-10},$$

Ryden finds

$$kT_{\text{nuc}} \approx 0.066 \text{ MeV},$$

so

$$T_{\text{nuc}} \approx 7.6 \times 10^8 \text{ K}, \qquad t_{\text{nuc}} \approx 200 \text{ s}.$$

This delay relative to freezeout is why neutron decay matters in setting the helium fraction. *Ryden, Chapter 9, pp. 14–16.* [oai$_c$*itation* : 10*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.10 Beyond deuterium

Once deuterium forms in significant quantity, strong-interaction reactions quickly build up

$$^3\mathrm{H}, \quad ^3\mathrm{He}, \quad ^4\mathrm{He}.$$

The key point is that once the bottleneck is passed, reactions toward ^{4}He are fast. Thus most surviving neutrons end up in ^{4}He. *Ryden, Chapter 9, pp. 16–18.* [oai$_c$*itation* : 11*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.11 Why nucleosynthesis stops at light nuclei

BBN does not efficiently produce heavy elements because of nuclear bottlenecks:

- there are no stable nuclei with $A = 5$,
- there are no stable nuclei with $A = 8$.

This prevents efficient fusion beyond ^{4}He, leaving only trace amounts of lithium and related nuclei. *Ryden, Chapter 9, pp. 17–20.* [oai$_c$*itation* : 12*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.12 Predicted primordial abundances

The main BBN products are:

- most baryons left as ^{1}H,
- about 24% by mass in ^{4}He,
- small leftovers of D and ^{3}He,
- trace ^{7}Li.

Thus BBN is efficient enough to make helium, but not efficient enough to process most baryons into heavier nuclei. *Ryden, Chapter 9, pp. 18–21.* [oai$_c$*itation* : 13*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.13 Dependence on the baryon-to-photon ratio η

The primordial abundances depend on η:

- larger η gives earlier, more efficient nucleosynthesis,
- this tends to increase ^{4}He,
- and decrease leftover D and ^{3}He.

Among the light elements, deuterium is the most sensitive to η, which is why it is the best baryometer. *Ryden, Chapter 9, pp. 19–21.* [oai$_c$*itation* : 14*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.14 Deuterium as a baryometer

The primordial deuterium abundance is commonly expressed as

$$\frac{D}{H}.$$

Observations of distant quasar absorption systems give

$$\left(\frac{D}{H}\right) \approx (2.53 \pm 0.04) \times 10^{-5},$$

which implies

$$\eta \approx (6.0 \pm 0.1) \times 10^{-10}.$$

This agrees well with the value inferred from the CMB, which is a major success of standard cosmology. *Ryden, Chapter 9, pp. 20–21.* [oai$_c$*itation* : 15*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.15 Baryon–antibaryon asymmetry

The baryon-to-photon ratio is tiny:

$$\eta \sim 6 \times 10^{-10}.$$

This suggests that in the early universe there was only a very slight excess of quarks over antiquarks. Most quark–antiquark pairs annihilated, leaving:

- a tiny residue of baryons,
- a huge sea of photons.

Thus the strong present-day asymmetry between baryons and antibaryons, and the large number of photons per baryon, are both consequences of a very small primordial quark–antiquark asymmetry. The exact origin of this asymmetry is not yet known. *Ryden, Chapter 9, pp. 22–24.* [oai$_c$*itation* : 16*RydenCh*9.*pdf*](*sediment* : //*file*$_0$00000009*f*2*c*720*c*95*bc*3*f*7*baa*1*e*12*d*7)

13.16 High-yield Chapter 9 formulas

$$\frac{n_n}{n_p} \approx e^{-Q_n/kT}$$

$$Q_n = (m_n - m_p)c^2 = 1.29 \text{ MeV}$$

$$f = e^{-t/\tau_n}$$

$$Y_{\max} = \frac{2f_n}{1+f_n}, \qquad f_n \equiv \frac{n_n}{n_p}$$

$$p + n \leftrightarrow D + \gamma$$

$$\frac{n_D}{n_n} = 1 \qquad \text{defines } T_{\text{nuc}}$$

13.17 Typical Chapter 9 exam asks

- Compute neutron-to-proton ratio at freezeout.
- Include neutron decay before nucleosynthesis.
- Estimate $Y_{\max}$ for helium.
- Explain the deuterium bottleneck.
- Explain why BBN stops at light nuclei.
- Explain why deuterium is the best baryometer.
- Interpret the meaning of the tiny baryon-to-photon ratio.

13.18 Chapter 9 traps / reminders

- Do not assume nucleosynthesis starts immediately at freeze-out.
- Do not forget neutron decay during the delay to t_{nuc}.
- Do not confuse ^{4}He fraction by mass with number fraction.
- The bottleneck is caused by photodissociation of deuterium, not by lack of neutrons alone.
- Heavier elements are not efficiently produced because of the lack of stable $A = 5$ and $A = 8$ nuclei.

14 Chapter 10 — Inflation

14.1 Why inflation was proposed

Standard Hot Big Bang cosmology explains much, but it leaves several puzzles:

- the **flatness problem**,
- the **horizon problem**,
- the **monopole problem**.

Inflation is a period of accelerated, approximately exponential expansion in the very early universe that resolves these problems. *Ryden, Chapter 10, pp. 1–9.* [oai$_c$*itation* : 17*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.2 Exponential expansion

During inflation,

$$a(t) \propto e^{Ht},$$

with approximately constant H.

If inflation lasts from t_i to t_f, then the number of e-foldings is

$$N = \ln\left(\frac{a_f}{a_i}\right).$$

Thus

$$\frac{a_f}{a_i} = e^N.$$

A modest increase in N produces an enormous increase in scale factor. *Ryden, Chapter 10, pp. 5–7.* [oai$_c$*itation* : 18*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.3 Flatness problem

Before inflation, if the universe is radiation-dominated, then

$$a \propto t^{1/2}, \qquad H \propto t^{-1}.$$

Since

$$|\Omega - 1| \propto \frac{1}{a^2 H^2},$$

it follows that before inflation

$$|\Omega - 1| \propto t.$$

Thus, without inflation, Ω must have been extraordinarily close to 1 at very early times in order to remain near 1 today.

Inflation solves this by driving Ω toward 1, making flatness a natural outcome rather than a fine-tuned initial condition. *Ryden, Chapter 10, Exercises 10.1 and surrounding discussion.* [oai$_c$*itation* : 19*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.4 Horizon problem

The CMB is nearly isotropic across regions of the sky that, in a non-inflationary model, were never in causal contact before recombination.

Inflation solves this by taking a tiny region that *was* in causal contact before inflation and stretching it to a size much larger than the observable universe. Thus apparently disconnected regions today can share a common pre-inflationary history. *Ryden, Chapter 10, context of inflation discussion.* [oai$_c$*itation* : 20*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.5 Monopole problem

Grand Unified Theories generically predict magnetic monopoles. Without inflation, their number density would be unacceptably large.

Inflation dilutes any relic number density exponentially:

$$n \propto a^{-3} \propto e^{-3N}.$$

Thus even a modest number of e-foldings can reduce monopole density to negligible levels. *Ryden, Chapter 10, Exercise 10.2 and discussion.* [oai$_c$*itation* : 21*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.6 Inflaton field

A common phenomenological description of inflation uses a scalar field ϕ, the **inflaton**, with potential $V(\phi)$.

If the field is spatially homogeneous, its energy density is

$$\varepsilon_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi),$$

and its pressure is

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

If the field evolves slowly enough that

$$\dot{\phi}^2 \ll V(\phi),$$

then

$$\varepsilon_\phi \approx V(\phi), \qquad P_\phi \approx -V(\phi),$$

so the inflaton behaves approximately like a cosmological constant:

$$w \approx -1.$$

This is the basic mechanism by which the inflaton can drive inflation. *Ryden, Chapter 10, pp. 1–3.* [oai$_c$*itation* : 22*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.7 Slow-roll / Hubble friction idea

The inflaton obeys an equation of motion analogous to a particle rolling down a potential with friction. The expansion of the universe provides a **Hubble friction** term, which can keep the field rolling slowly.

Inflation occurs when:

- the potential $V(\phi)$ is large enough to dominate the energy density,
- the slope $dV/d\phi$ is shallow enough that the field evolves slowly.

This is the physical content of the slow-roll idea. *Ryden, Chapter 10, pp. 2–5.* [oai$_c$*itation* : 23*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.8 False vacuum and true vacuum

A useful picture is a potential with:

- a broad, shallow plateau near $\phi \approx 0$,
- a true minimum at $\phi = \phi_0$.

If the field starts on the plateau, it is in a metastable **false vacuum** state. While it slowly rolls toward the true vacuum, the nearly constant potential energy drives inflation.

The broader, higher, and flatter the plateau, the longer inflation lasts and the more e-foldings are produced. *Ryden, Chapter 10, pp. 4–6.* [oai$_c$*itation* : 24*Ryden*$_C$*h*10.*pdf*](*sediment* : *//file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.9 When inflation starts

Inflation begins when the inflaton energy density overtakes the radiation density:

$$V_0 \gtrsim \alpha T^4.$$

Thus the characteristic start temperature is set by

$$T_i \sim \left(\frac{V_0}{\alpha}\right)^{1/4}.$$

The corresponding start time depends on the thermal history before inflation. *Ryden, Chapter 10, pp. 5–6.* [oai$_c$*itation* : 25*Ryden*$_C$*h*10.*pdf*](*sediment* : //*file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.10 Reheating

Inflation dilutes not only monopoles, but also photons and all other particles. Thus inflation must end with a process that repopulates the universe with radiation.

As the inflaton rolls to the true vacuum and oscillates around the minimum, its energy is transferred to ordinary particles. This is **reheating**. The latent energy of the false vacuum becomes the thermal bath of the post-inflation universe. *Ryden, Chapter 10, pp. 6–7.* [oai$_c$*itation* : 26*Ryden*$_C$*h*10.*pdf*](*sediment* : //*file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.11 Why reheating is essential

Without reheating, inflation would leave the universe extremely cold and nearly empty. Reheating ensures that after inflation the universe again contains a hot bath of relativistic particles, allowing the usual Hot Big Bang evolution to resume.

Thus inflation does not replace the Hot Big Bang; it modifies the very early initial conditions for it. *Ryden, Chapter 10, pp. 6–7.* [oai$_c$*itation* : 27*Ryden*$_C$*h*10.*pdf*](*sediment* : //*file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.12 Quantum fluctuations as the seeds of structure

Inflation smooths out classical inhomogeneities extremely efficiently. If that were the whole story, the universe would be too smooth.

However, quantum fluctuations in the inflaton field are stretched by inflation from microscopic to macroscopic scales. These stretched fluctuations become the seeds for density perturbations, which later grow into galaxies, clusters, and large-scale structure.

Thus inflation explains not only the large-scale smoothness of the universe, but also the origin of its small inhomogeneities. *Ryden, Chapter 10, pp. 7–8.* [oai$_c$*itation* : 28*Ryden*$_C$*h*10.*pdf*](*sediment* : //*file*$_0$0000000*e*45471*fd*88950*a*6*ffc*51*dad*6)

14.13 Late-time false vacuum decay

The chapter also considers the idea that the current acceleration could be due to a false vacuum that may someday decay. If a false vacuum decays to radiation, the universe reheats and then resumes ordinary radiation- and matter-dominated evolution afterward.

This kind of problem mixes:

- vacuum-dominated exponential expansion,
- reheating temperature from $\varepsilon_r = \alpha T^4$,
- matter redshifting as a^{-3},
- radiation redshifting as a^{-4}.

14.14 High-yield Chapter 10 formulas

$$a(t) \propto e^{Ht}$$

$$N = \ln\left(\frac{a_f}{a_i}\right)$$

$$n \propto e^{-3N} \qquad \text{(relic dilution)}$$

$$|\Omega - 1| \propto t \qquad \text{before inflation in radiation domination}$$

$$\varepsilon_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Slow-roll limit:

$$\dot{\phi}^2 \ll V(\phi) \quad \Rightarrow \quad \varepsilon_\phi \approx V(\phi), \quad P_\phi \approx -V(\phi)$$

14.15 Typical Chapter 10 exam asks

- Explain the flatness, horizon, and monopole problems.
- Estimate how close Ω must be to 1 at early times without inflation.
- Estimate how many e-foldings are needed to dilute a relic density.
- Explain how the inflaton field can mimic a cosmological constant.
- Explain the meaning of false vacuum, true vacuum, and reheating.
- Explain how inflation can both smooth the universe and generate structure.

14.16 Chapter 10 traps / reminders

- Inflation does not mean the universe was always exponentially expanding; it is a temporary early phase.
- Inflation solves the horizon problem by enlarging a previously causally connected patch.
- Inflation dilutes *all* relics, not just monopoles.
- Reheating is essential; otherwise the universe would be too empty and cold after inflation.
- The false vacuum is metastable, not the true minimum.
- Inflation smooths classical perturbations but stretches quantum fluctuations into cosmological seeds.

15 Cross-Chapter Concept Web

15.1 Redshift, scale factor, and wavelength

One of the most important chains in the course is

$$1 + z = \frac{a_0}{a_{\rm em}} = \frac{\lambda_{\rm obs}}{\lambda_{\rm em}}.$$

This connects:

- Chapter 2: redshift as an observable,
- Chapter 3: the scale factor in the Robertson–Walker metric,
- Chapter 6: cosmological distance measures,
- later chapters: thermal history through $T \propto a^{-1}$.

Thus whenever a problem gives z, you should immediately think:

$$z \to a_{\rm em} \to \lambda_{\rm obs} \to \text{cosmic epoch}.$$

15.2 Equation of state, density evolution, and expansion law

The core dynamical chain is

$$P = w\varepsilon \quad \Rightarrow \quad \varepsilon \propto a^{-3(1+w)} \quad \Rightarrow \quad H(a) \quad \Rightarrow \quad a(t).$$

This links:

- Chapter 4: continuity equation and equation of state,
- Chapter 5: model universes,
- Chapter 10: inflation as an effective $w \approx -1$ phase.

Important special cases:

$$w = 0 \Rightarrow \varepsilon_m \propto a^{-3}, \quad a(t) \propto t^{2/3}$$

$$w = \frac{1}{3} \Rightarrow \varepsilon_r \propto a^{-4}, \quad a(t) \propto t^{1/2}$$

$$w = -1 \Rightarrow \varepsilon_\Lambda = \text{constant}, \quad a(t) \propto e^{Ht}$$

15.3 Cosmic microwave background, recombination, and last scattering

The CMB, recombination, and last scattering belong to one unified story:

- before recombination, the universe is ionized and photons scatter efficiently,
- as recombination occurs, the free-electron density drops,
- photons decouple and begin to free-stream,
- we observe those photons today as the CMB.

This links:

- Chapter 2: the CMB as evidence for the Hot Big Bang,
- Chapter 8: the microphysics of recombination and decoupling,
- Chapter 6: distance interpretation of the last scattering surface.

15.4 BBN, baryon-to-photon ratio, and deuterium

The baryon-to-photon ratio

$$\eta = \frac{n_{\text{bary}}}{n_\gamma}$$

appears in both:

- Chapter 8: recombination through the Saha equation,
- Chapter 9: nucleosynthesis through deuterium synthesis and light-element abundances.

Deuterium is especially important because its primordial abundance depends strongly on η, making it the best baryometer.

Thus the same quantity η helps control both:

- the thermal ionization history,
- the light-element yields of BBN.

15.5 Dark matter from galaxies, clusters, and lensing

Dark matter is not inferred from a single observation. The same conclusion emerges from several logically distinct methods:

- galaxy rotation curves show $M(R)$ continues growing beyond the luminous matter,
- virial analyses of galaxies and clusters show mass greatly exceeds luminous matter,
- gravitational lensing directly traces total gravitating mass,
- cosmological matter budgets require more matter than baryons alone.

This is why the dark matter case is strong: the evidence is redundant and mutually reinforcing.

15.6 Inflation and the early-universe problems

Inflation ties together several otherwise separate problems:

- flatness problem,
- horizon problem,
- monopole problem,
- origin of primordial fluctuations.

It also links to later cosmology because:

- it sets the initial conditions for the Hot Big Bang,
- reheating restores the radiation bath needed for standard thermal history,
- quantum fluctuations stretched by inflation seed the inhomogeneities seen later in the CMB and structure formation.

15.7 Distances and observables

A very important practical chain is:

- luminosity L + flux $f \to d_L$,
- size ℓ + angular size $\delta\theta \to d_A$,
- expansion history $\to d_P$,
- then

$$d_L = (1+z)d_P, \qquad d_A = \frac{d_P}{1+z}.$$

Thus observational astronomy and cosmological geometry meet in Chapter 6.

15.8 Thermal history overview

The thermal history chain is:

very early hot universe $\to$ weak freezeout $\to$ BBN $\to$ recombination $\to$ last

This is one of the best mental maps for the final exam.

16 Comparison Tables

16.1 Matter vs radiation vs cosmological constant

16.2 Proper distance vs luminosity distance vs angular-diameter distance

And the key relations:

$$d_L = (1+z)d_P, \qquad d_A = \frac{d_P}{1+z}, \qquad d_A = \frac{d_L}{(1+z)^2}.$$

16.3 Recombination vs nucleosynthesis

16.4 Evidence for dark matter

16.5 Inflation: problems and solutions

17 Common Mistakes and Trap Sheet

17.1 Sign mistakes

- The curvature term in the Friedmann equation is

$$-\frac{kc^2}{R_0^2 a^2},$$

not $+\frac{kc^2}{R_0^2 a^2}$.

Component	w	Scaling with a	Pressure	Effect on expansion
Matter	0	a^{-3}	negligible	decelerates
Radiation	$\frac{1}{3}$	a^{-4}	positive	decelerates more strongly
Λ	-1	constant	negative	accelerates

Table 1: Comparison of matter, radiation, and cosmological constant.

Distance	Symbol	Used with	Relation
Proper distance	d_P	geometry / spacelike separation	$d_P = c\int dt/a(t)$
Luminosity distance	d_L	flux and luminosity	$f = L/(4\pi d_L^2)$
Angular-diameter distance	d_A	size and angle	$d_A = \ell/\delta\theta$

Table 2: Comparison of the main cosmological distance measures.

- The acceleration equation depends on

$$\varepsilon + 3P,$$

not just $\varepsilon + P$.
- For vacuum energy,

$$P_\Lambda = -\varepsilon_\Lambda,$$

not $+\varepsilon_\Lambda$.

17.2 Convention mistakes

- $a_0 = 1$ is a convention.
- $k = +1, 0, -1$ labels the sign of curvature, not directly the fate by itself.
- $F(> E_0)$ means the fraction of photons with $E > E_0$.

17.3 Distance confusion

- d_P, d_L, and d_A are not interchangeable.
- Luminosity distance is used with L and f.
- Angular-diameter distance is used with ℓ and $\delta\theta$.
- Remember:

$$d_A = \frac{d_L}{(1+z)^2}.$$

17.4 Low-redshift misuse

- Do not use

$$z \approx \frac{v}{c}$$

unless the problem is explicitly low-z.
- Likewise, do not extend simple Hubble-law intuition blindly to large redshift without checking context.

17.5 Thermal-history confusion

- Recombination and decoupling are related but not identical.
- Recombination does not happen at $kT = 13.6$ eV.
- Nucleosynthesis does not start immediately at weak freeze-out.
- Do not forget neutron decay between freezeout and deuterium synthesis.

17.6 Dark matter traps

- Flat rotation curves imply $M(R) \propto R$, not constant enclosed mass.
- Virial estimates assume approximate equilibrium.
- Lensing measures total mass, not visible mass only.

17.7 Inflation traps

- Inflation is a temporary early phase, not the whole expansion history.
- Inflation dilutes all relics, so reheating is required afterward.
- Inflation smooths classical perturbations but amplifies quantum fluctuations into cosmological seeds.

18 One-Page Recovery Checklist

18.1 If I do not know how to start

Use the following emergency procedure:

1. Identify the topic:
   - redshift / observation,
   - geometry / FRW,
   - Friedmann dynamics,
   - distances / magnitudes,
   - dark matter / virial / lensing,
   - recombination / BBN,
   - inflation.
2. Write the governing equation immediately.
3. Define every variable.
4. Decide what approximation applies.

18.2 Typical starting equations by topic

- Redshift:

$$1 + z = \frac{a_0}{a_{\text{em}}}$$

- Hubble law:

$$v = H_0 r$$

- Friedmann:

$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2} + \frac{\Lambda c^2}{3}$$

- Continuity:

$$\dot{\varepsilon} = -3H(\varepsilon + P)$$

- Distance modulus:

$$m - M = 5\log_{10}\left(\frac{d_L}{10\text{ pc}}\right)$$

- Flux/luminosity:

$$f = \frac{L}{4\pi d_L^2}$$

Feature	Recombination	Nucleosynthesis
Main process	$p + e^- \leftrightarrow H + \gamma$	$p + n \leftrightarrow D + \gamma$, then light nuclei
Energy scale	eV	MeV
Typical temperature	$\sim 10^3$ K	$\sim 10^9$ K
Key equation	Saha equation	Saha-like deuterium equilibrium + freezeout physics
Main observable	CMB last scattering	primordial light-element abundances
Key parameter	η	η

Table 3: Comparison of recombination and Big Bang nucleosynthesis.

Method	What is measured	Why it implies dark matter
Rotation curves	$v(R)$	$M(R)$ keeps growing where light does not
Virial theorem	σ_r, r_h	total mass exceeds visible matter
Gravitational lensing	light deflection	lensing traces total mass, not light
Cosmological budget	Ω_b, Ω_m	baryons are not enough to make all matter

Table 4: Main observational lines of evidence for dark matter.

- Angular size:
$$d_A = \frac{\ell}{\delta\theta}$$
- Rotation curve mass:
$$M(R) = \frac{v^2 R}{G}$$
- Virial mass:
$$M \sim \frac{\langle v^2 \rangle r_h}{\alpha G}$$
- Lensing:
$$\alpha = \frac{4GM}{bc^2}$$
- Saha-equilibrium style:
$$\frac{X^2}{1-X} = S(T, \eta)$$
- BBN helium estimate:
$$Y_{\max} = \frac{2 f_n}{1 + f_n}$$
- Inflation:
$$a \propto e^{Ht}, \qquad N = \ln\left(\frac{a_f}{a_i}\right)$$

18.3 Unit and sanity checks

Before finalizing an answer:

- Does the answer have the correct units?
- Is the sign physically sensible?
- Does the scaling make sense?
- Does the result reduce correctly in a limiting case?

18.4 Common sanity checks

- Higher redshift should mean smaller $a_{\rm em}$.
- Radiation should redshift faster than matter.
- A flat rotation curve should imply increasing enclosed mass.
- If Λ dominates, the expansion should accelerate.
- If a result says recombination happens at $\sim 10^5$ K, something is wrong.

Problem	Inflationary resolution
Flatness problem	drives Ω toward 1
Horizon problem	stretches one causal patch to enormous size
Monopole problem	dilutes relic number density by e^{-3N}
Origin of structure	stretches quantum fluctuations to macroscopic scales

Table 5: Problems addressed by inflation.