Quantum Mechanics

Master Revision Guide

Foundations · Derivations · Physical Intuition · Problem Solving

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1 Mathematical Foundations

1.1 Hilbert Space and Basis Expansions

A pure quantum state is a normalized ray in a complex Hilbert space. Writing it in an orthonormal basis,

$$|\psi\rangle = \sum_n c_n |n\rangle, \qquad c_n = \langle n|\psi|n|\psi\rangle, \qquad \sum_n |c_n|^2 = 1.$$

The state $e^{i\phi}|\psi\rangle$ represents the same physical ray, but relative phases in superpositions produce interference. Orthonormality and completeness are

$$\langle m|n|m|n\rangle = \delta_{mn}, \qquad \boxed{\sum_n |n\rangle\langle n| = I}.$$

An inner product $\langle\phi|\psi|\phi|\psi\rangle$ is a complex number; an outer product $|\psi\rangle\langle\phi|$ is an operator. For $|\psi\rangle = (a,b)^T$, the bra is (a^*, b^*).

1.2 Operators and Spectral Decomposition

Physical observables correspond to self-adjoint operators $A = A^\dagger$. Their spectral projectors define measurement outcomes. In a discrete nondegenerate spectrum,

$$A|n\rangle = a_n|n\rangle, \quad A = \sum_n a_n |n\rangle\langle n|.$$

The expectation and variance are

$$\boxed{\langle A\rangle = \langle\psi|A|\psi\rangle}, \quad (\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2.$$

Unitary operators $U^\dagger U = I$ preserve inner products and implement changes of basis or physical evolution. Under an active unitary state transformation $|\psi\rangle \to U|\psi\rangle$, an equivalent passive operator representation is $A \to U^\dagger A U$.

1.3 Commutators and Useful Identities

$$[A,B] = AB - BA, \qquad \{A,B\} = AB + BA.$$

$$[A,BC] = [A,B]C + B[A,C], \quad [AB,C] = A[B,C] + [A,C]B.$$

The canonical commutator implies

$$\boxed{[x,p] = i\hbar}, \quad [x,p^n] = i\hbar n p^{n-1}, \quad [p,f(x)] = -i\hbar f'(x).$$

The Baker–Campbell–Hausdorff conjugation series,

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots,$$

solves operator translation and time-evolution problems when nested commutators are tractable. For $T(a) = e^{-iap/\hbar}$, $T(a)^\dagger x T(a) = x + a$.

Core Result. A large fraction of quantum calculations reduce to choosing a convenient basis, inserting completeness, and computing a few commutators.

2 Measurement and Uncertainty

2.1 Projective Measurement

For the spectral projector P_a onto the eigenspace with eigenvalue a,

$$\boxed{p(a) = \langle\psi|P_a|\psi\rangle}, \qquad |\psi_a\rangle = \frac{P_a|\psi\rangle}{\sqrt{p(a)}}.$$

In a nondegenerate basis, $p(a) = |\langle a|\psi|a|\psi\rangle|^2$. When the eigenvalue is degenerate, the post-measurement state is projected onto the *whole* eigenspace. Probabilities are squared moduli of amplitudes; amplitudes themselves are complex.

2.2 Uncertainty Relation and Derivation

Let $\delta A = A - \langle A\rangle$ and $\delta B = B - \langle B\rangle$. By Cauchy–Schwarz applied to $\delta A|\psi\rangle$ and $\delta B|\psi\rangle$,

$$(\Delta A)^2(\Delta B)^2 \geq |\langle\delta A\delta B\rangle|^2.$$

The imaginary part is fixed by the commutator, leading to

$$\boxed{\Delta A\,\Delta B \geq \frac{1}{2}|\langle[A,B]\rangle|}, \qquad \Delta x \Delta p \geq \frac{\hbar}{2}.$$

Retaining the real part yields Schrödinger–Robertson:

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle[A,B]\rangle|^2 + \frac{1}{4}|\langle\{\delta A, \delta B\}\rangle|^2.$$

Exam Insight. A nonzero operator commutator need not yield a positive Robertson lower bound in every state: its expectation may vanish.

3 Qubits and Pauli Algebra

3.1 Bloch-Sphere State

Every normalized pure qubit, up to global phase, can be written

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle .$$

The Pauli matrices are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They obey

$$\boxed{\sigma_i\sigma_j = \delta_{ij}I + i\epsilon_{ijk}\sigma_k},$$

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I.$$

Consequently, for real vectors $\mathbf{a}, \mathbf{b}$,

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})I + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}.$$

3.2 Rotations and Density Matrices

Because $(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = I$ for unit $\mathbf{n}$,

$$\boxed{e^{-i\theta\mathbf{n}\cdot\boldsymbol{\sigma}/2} = I\cos\frac{\theta}{2} - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\frac{\theta}{2}}.$$

A 2π spin-1/2 rotation gives $-I$, physically important when compared to an unrotated branch by interference. A general qubit density operator is

$$\rho = \tfrac{1}{2}(I + \mathbf{r}\cdot\boldsymbol{\sigma}), \quad \operatorname{Tr}\rho^2 = \tfrac{1}{2}(1 + |\mathbf{r}|^2).$$

Pure states have $|\mathbf{r}| = 1$; mixed states have $|\mathbf{r}| < 1$.

4 Composite Systems, SWAP, and Entanglement

4.1 Tensor Products

For two qubits the Hilbert space is $\mathbb{C}^2 \otimes \mathbb{C}^2$ with ordered basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Product states factor into single-system kets; entangled pure states do not.

4.2 SWAP Matrix and Projectors

The exchange operator $S|ab\rangle = |ba\rangle$ has matrix

$$\boxed{S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}.$$

Its Pauli representation is

$$\boxed{S = \frac{1}{2}(I\otimes I + X\otimes X + Y\otimes Y + Z\otimes Z)}.$$

Since $S^2 = I$, its eigenvalues are ± 1. The symmetric (triplet) subspace has three dimensions, while the antisymmetric singlet state is

$$|s\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

Corresponding projectors are $P_+ = (I + S)/2$ and $P_- = (I - S)/2$.

4.3 Exchange Evolution

For $H = JS$, use $S^2 = I$ to sum the exponential:

$$\boxed{e^{-iJSt/\hbar} = I\cos(Jt/\hbar) - iS\sin(Jt/\hbar)}.$$

In particular,

$$U(t)|10\rangle = \cos(Jt/\hbar)|10\rangle - i\sin(Jt/\hbar)|01\rangle .$$

Complete exchange occurs at $t = \pi\hbar/(2|J|)$, modulo an overall phase.

4.4 Partial Trace and Entanglement

For $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, the full state $\rho = |\Phi^+\rangle\langle\Phi^+|$ is pure, but $\rho_A = \operatorname{Tr}_B\rho = I/2$. The subsystem von Neumann entropy is $S_A = -\operatorname{Tr}(\rho_A\ln\rho_A) = \ln 2$.

5 Schrödinger and Heisenberg Pictures

5.1 Unitary Time Evolution

The Schrödinger equation is

$$\boxed{i\hbar\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle}.$$

For constant H, $U(t) = e^{-iHt/\hbar}$. Expand an initial state in energy eigenvectors:

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|n\rangle .$$

Energy eigenstates only acquire phases; superpositions generally change relative phase.

5.2 Heisenberg Equation

For $A_H = U^\dagger A_S U$,

$$\boxed{\frac{dA_H}{dt} = \frac{i}{\hbar}[H_H, A_H] + \left(\frac{\partial A}{\partial t}\right)_H}.$$

For $H = p^2/(2m) + V(x)$ this implies Ehrenfest's theorem:

$$\frac{d\langle x\rangle}{dt} = \frac{\langle p\rangle}{m}, \quad \frac{d\langle p\rangle}{dt} = -\langle V'(x)\rangle.$$

A time-independent observable is conserved when $[H, A] = 0$.

5.3 Any Two-Level Hamiltonian

Write $H = h_0 I + \mathbf{h}\cdot\boldsymbol{\sigma}$. Its levels are $h_0 \pm |\mathbf{h}|$ and its exact propagator is

$$U(t) = e^{-ih_0t/\hbar}\left[I\cos\frac{|\mathbf{h}|t}{\hbar} - i\frac{\mathbf{h}\cdot\boldsymbol{\sigma}}{|\mathbf{h}|}\sin\frac{|\mathbf{h}|t}{\hbar}\right].$$

This one identity solves precession, coupled spin, and avoided-crossing dynamics.

6 Angular Momentum and Spin

6.1 SU(2) Algebra

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \qquad J_\pm = J_x \pm iJ_y.$$

The states $|jm\rangle$ satisfy

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle, \quad J_z|jm\rangle = \hbar m|jm\rangle.$$

The ladder action is

$$\boxed{J_\pm|jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle}.$$

Here $m = -j, -j+1, \ldots, j$.

6.2 Spin-One Representation

In the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$,

$$S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},$$

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}.$$

For $H = \omega S_z$, each magnetic sublevel acquires $e^{-im\omega t}$.

6.3 Addition of Angular Momentum

For $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$, allowed j values range from $|j_1 - j_2|$ to $j_1 + j_2$ in integer steps. Coupled states and product states are related by Clebsch–Gordan coefficients:

$$|jm\rangle = \sum_{m_1,m_2} C^{jm}_{j_1m_1,j_2m_2}|j_1m_1\rangle|j_2m_2\rangle.$$

For $1/2 \otimes 1/2 = 1 \oplus 0$, the triplets are $|\uparrow\uparrow\rangle$, $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$, $|\downarrow\downarrow\rangle$; the singlet has the minus sign.

6.4 Spin-Orbit Coupling

Because $J^2 = L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S}$,

$$\boxed{\mathbf{L}\cdot\mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2)}.$$

For $H_{SO} = \lambda\mathbf{L}\cdot\mathbf{S}$,

$$E_j = \frac{\lambda\hbar^2}{2}\{j(j+1) - l(l+1) - s(s+1)\}.$$

With $s = 1/2$, $j = l \pm 1/2$ gives $E_{l+1/2} = \lambda\hbar^2 l/2$ and $E_{l-1/2} = -\lambda\hbar^2(l+1)/2$ (for $l \geq 1$).

Exam Insight. If the Hamiltonian contains a scalar product of angular momenta, try the total-J^2 identity before building a matrix.

7 Identical Particles and Second Quantization

Bosonic two-particle wavefunctions are symmetric under exchange; fermionic wavefunctions are antisymmetric. For orthonormal orbitals a, b,

$$\psi_\pm(x_1, x_2) = \frac{\phi_a(x_1)\phi_b(x_2) \pm \phi_b(x_1)\phi_a(x_2)}{\sqrt{2}}.$$

The antisymmetric combination vanishes if both fermions occupy the same orbital, yielding Pauli exclusion. Fermionic N-particle states can be written as Slater determinants.

7.1 Creation and Annihilation Algebra

Bosons satisfy $[a_i, a_j^\dagger] = \delta_{ij}$; fermions satisfy $\{c_i, c_j^\dagger\} = \delta_{ij}$. Number operators are $n_i = a_i^\dagger a_i$ or $c_i^\dagger c_i$. Fermionic occupations are $0, 1$ while bosonic occupations are arbitrary nonnegative integers.

7.2 Schwinger Bosons

Using two independent boson modes,

$$J_+ = \hbar a^\dagger b, \quad J_- = \hbar b^\dagger a, \quad J_z = \frac{\hbar}{2}(a^\dagger a - b^\dagger b).$$

At fixed total occupation $N = n_a + n_b = 2j$, the state $|jm\rangle$ corresponds to $|n_a = j+m, n_b = j-m\rangle$ and $J^2 = \hbar^2 N(N+2)/4$.

8 Harmonic Oscillator and Coherent States

8.1 Ladder Operators

For $H = p^2/(2m) + m\omega^2 x^2/2$, define

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{ip}{\sqrt{2m\hbar\omega}}, \quad [a, a^\dagger] = 1.$$

Then

$$\boxed{H = \hbar\omega(a^\dagger a + \tfrac{1}{2})}, \qquad E_n = \hbar\omega(n + \tfrac{1}{2}).$$

Their action is $a|n\rangle = \sqrt{n}\,|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$. The ground state solves $a|0\rangle = 0$:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/(2\hbar)}.$$

8.2 Selection Rules and Moments

Express

$$x = x_0(a + a^\dagger), \quad x_0 = \sqrt{\frac{\hbar}{2m\omega}}, \quad p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a).$$

Therefore x couples $\Delta n = \pm 1$, while x^2 couples $\Delta n = 0, \pm 2$. In a number eigenstate,

$$\langle x^2\rangle_n = \frac{\hbar}{m\omega}(n + \tfrac{1}{2}), \quad \langle p^2\rangle_n = m\hbar\omega(n + \tfrac{1}{2}).$$

The kinetic and potential expectation values are each $E_n/2$.

8.3 Heisenberg Motion

The commutator $[H, a] = -\hbar\omega a$ gives $a(t) = a(0)e^{-i\omega t}$. Hence

$$x(t) = x(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t.$$

8.4 Coherent States

Coherent states satisfy $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and expand as

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,.$$

They have Poisson number statistics, $P(n) = e^{-|\alpha|^2}|\alpha|^{2n}/n!$, and saturate $\Delta x\Delta p = \hbar/2$. Their mean position follows classical oscillator motion.

9 Wavefunctions, Current, and Wavepackets

9.1 Position-Space Schrödinger Equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + V(x,t)\psi, \qquad \rho = |\psi|^2.$$

For real V, multiply the equation by ψ^*, subtract its complex conjugate multiplied by ψ, and rearrange to obtain

$$\boxed{\partial_t\rho + \partial_x j = 0}, \qquad \boxed{j = \frac{\hbar}{m}\,\mathrm{Im}(\psi^*\partial_x\psi)}.$$

Integrating over all space yields conservation of normalization when the net boundary current vanishes.

9.2 Free Gaussian Packet

Take

$$\psi(x,0) = (\pi\sigma^2)^{-1/4}e^{-x^2/(2\sigma^2)}, \quad \tau = \frac{m\sigma^2}{\hbar}.$$

The free evolution is

$$\psi(x,t) = \frac{(\pi\sigma^2)^{-1/4}}{\sqrt{1+it/\tau}}\exp\left[-\frac{x^2}{2\sigma^2(1+it/\tau)}\right].$$

Its density is Gaussian with width parameter

$$\boxed{\sigma_t = \sigma\sqrt{1+(t/\tau)^2}}, \qquad \Delta x(t) = \sigma_t/\sqrt{2}.$$

Since $\Delta p = \hbar/(\sqrt{2}\sigma)$, momentum components have different velocities p/m, causing spreading without loss of integrated probability.

10 Propagators and Path Integrals

10.1 The Propagator

Define $K(x_f,t_f;x_i,t_i) = \langle x_f|\,U(t_f,t_i)\,|x_i\rangle$. Then

$$\psi(x_f,t_f) = \int dx_i\,K(x_f,t_f;x_i,t_i)\psi(x_i,t_i).$$

Inserting identity at an intermediate time gives the composition law

$$K_{fi} = \int dx\,K(x_f,t_f;x,t)K(x,t;x_i,t_i).$$

For a free particle over time $t > 0$,

$$\boxed{K_0 = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im(x_f-x_i)^2}{2\hbar t}\right]}.$$

10.2 Time Slicing and the Action

Split time into N intervals of length ϵ, use the Trotter product, and insert $N-1$ position resolutions of identity. Each factor contains the short-time classical action. In the continuum limit,

$$\boxed{K = \int_{x_i}^{x_f}\mathcal{D}x(t)\,e^{iS[x]/\hbar}}, \quad S[x] = \int dt\,\left(\tfrac{1}{2}m\dot{x}^2 - V(x)\right).$$

When actions vary rapidly relative to $\hbar$, stationary-phase trajectories satisfy $\delta S = 0$, reproducing classical Euler–Lagrange dynamics.

10.3 Imaginary Time and Ground-State Projection

Under $t = -i\tau$, real-time phases become Euclidean weights $e^{-S_E/\hbar}$ where $S_E = \int d\tau(\frac{1}{2}m\dot{x}^2 + V)$. The operator $e^{-H\tau/\hbar}$ exponentially suppresses higher-energy components in a state (after normalization). This underlies imaginary-time propagation, thermal path integrals, and quantum Monte Carlo.

11 WKB, Tunneling, and Instantons

11.1 Semiclassical WKB Derivation

For $-\hbar^2\psi''/(2m) + V\psi = E\psi$, use $\psi = Ae^{iS/\hbar}$. At leading order $S'^2 = 2m(E-V)$. Let $p(x) = \sqrt{2m(E-V)}$ in the allowed region. The WKB solutions are

$$\psi \approx \frac{C_+e^{\frac{i}{\hbar}\int^x p\,dx'} + C_-e^{-\frac{i}{\hbar}\int^x p\,dx'}}{\sqrt{p(x)}}.$$

In forbidden regions use $\kappa = \sqrt{2m(V-E)}$ to get decaying or growing exponentials. Across a smooth barrier,

$$\boxed{T \sim \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V-E)}\,dx\right]}.$$

The two-turning-point Bohr–Sommerfeld condition is $\int_{x_1}^{x_2} p\,dx = \pi\hbar(n+1/2)$.

11.2 Double-Well Effective Model

For localized states $|L\rangle, |R\rangle$, write

$$H_{\text{eff}} = \begin{pmatrix} E_0 & -\Delta_t \\ -\Delta_t & E_0 \end{pmatrix}.$$

Symmetric and antisymmetric combinations differ in energy by $2|\Delta_t|$. In imaginary time, an instanton is a Euclidean classical trajectory connecting the minima; its leading tunneling contribution scales as $e^{-S_{\text{inst}}/\hbar}$.

11.3 Worldline Kinks

In a discretized spin imaginary-time path integral, changes of spin orientation between slices are worldline kinks. Their appearance reflects tunneling in the underlying quantum problem, but the numerical kink density is sensitive to time-slice discretization and should not automatically be interpreted as a universal order parameter.

12 Gauge Invariance and Magnetic Fields

12.1 Minimal Coupling and Covariance

For charge q,

$$\boxed{H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\Phi}, \qquad \boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}.$$

Under $\mathbf{A} \to \mathbf{A} + \nabla\chi$ and $\Phi \to \Phi - \partial_t\chi$, the wavefunction transforms as $\psi \to e^{iq\chi/\hbar}\psi$. Defining $\mathbf{D} = \nabla - iq\mathbf{A}/\hbar$ ensures $\mathbf{D}'\psi' = e^{iq\chi/\hbar}\mathbf{D}\psi$.

12.2 Aharonov–Bohm Flux

Around a closed loop the phase difference is

$$\boxed{\Delta\phi_{AB} = \frac{q}{\hbar}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{q\Phi_B}{\hbar}}.$$

Interference can depend on enclosed magnetic flux even when the particle traverses a field-free region. The flux period for a charge of magnitude $|q|$ is $\Phi_0 = h/|q|$.

12.3 Flux-Threaded Ring

For a ring of radius R and signed dimensionless flux $\varphi = q\Phi_B/h$,

$$H = \epsilon(-i\partial_\theta - \varphi)^2, \quad \epsilon = \frac{\hbar^2}{2mR^2}.$$

Eigenstates $e^{in\theta}/\sqrt{2\pi}$ have

$$\boxed{E_n = \epsilon(n-\varphi)^2}.$$

At $\varphi = 1/2$, $|0\rangle, |1\rangle$ are degenerate. A perturbation $V_0\cos\theta$ has matrix elements

$$\langle m|\, V\,|n\rangle = \frac{V_0}{2}(\delta_{m,n+1} + \delta_{m,n-1}),$$

so it mixes $\Delta n = \pm 1$. Restricted to the doublet, $H_{\text{eff}} = \epsilon I/4 + (V_0/2)\sigma_x$, opening gap $|V_0|$. With $\varphi = 1/2 + \delta$,

$$E_\pm = \epsilon(1/4 + \delta^2) \pm \sqrt{(\epsilon\delta)^2 + (V_0/2)^2}.$$

12.4 Cyclotron Motion and Landau Levels

For $\mathbf{B} = B\hat{z}$, the mechanical momenta satisfy $[\pi_x, \pi_y] = i\hbar qB$. For $H = (\pi_x^2 + \pi_y^2)/(2m)$,

$$\dot{\pi}_x = \frac{qB}{m}\pi_y, \quad \dot{\pi}_y = -\frac{qB}{m}\pi_x.$$

Thus $\omega_c = qB/m$ is the signed cyclotron frequency. The corresponding Landau levels are $E_n = \hbar|\omega_c|(n + 1/2)$.

13 Time-Independent Perturbation Theory

13.1 Nondegenerate Expansion

For $H = H_0 + \lambda V$, expand $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots$. Matching powers of λ gives

$$\boxed{E_n^{(1)} = \langle n|\, V\,|n\rangle},$$

$$\boxed{\left|n^{(1)}\right\rangle = \sum_{m\neq n} \frac{\langle m|\, V\,|n\rangle}{E_n^{(0)} - E_m^{(0)}}\,|m\rangle},$$

$$\boxed{E_n^{(2)} = \sum_{m\neq n} \frac{|\,\langle m|\, V\,|n\rangle\,|^2}{E_n^{(0)} - E_m^{(0)}}}.$$

For the nondegenerate ground state all denominators in the second-order correction are negative, so $E_0^{(2)} \le 0$.

13.2 Oscillator with a Linear Perturbation

Take $H = H_0 + \lambda x$. Because x is odd under parity, $\langle n|\, x\,|n\rangle = 0$. Only $n \pm 1$ contribute to the second-order sum:

$$E_n^{(2)} = x_0^2\left(\frac{n}{\hbar\omega} - \frac{n+1}{\hbar\omega}\right) = -\frac{1}{2m\omega^2}.$$

Completing the square yields the exact answer

$$\boxed{E_n = \hbar\omega(n + 1/2) - \frac{\lambda^2}{2m\omega^2}}.$$

Thus all higher energy corrections vanish.

13.3 Degenerate Subspaces

For degeneracy, diagonalize the matrix $W_{ab} = \langle a|\, V\,|b\rangle$ restricted to the degenerate subspace. The nondegenerate denominators fail at exact degeneracy. For a generic coupled doublet,

$$H = \begin{pmatrix} E_1 & v \\ v^* & E_2 \end{pmatrix},$$

$$\boxed{E_\pm = \frac{E_1 + E_2}{2} \pm \sqrt{\left(\frac{E_1 - E_2}{2}\right)^2 + |v|^2}}.$$

At resonance the avoided-crossing gap is $2|v|$.

Exam Insight. Before evaluating any perturbative sum, use symmetry and ladder-operator selection rules to eliminate forbidden matrix elements.

14 Time-Dependent Perturbation Theory

14.1 Interaction Picture and Dyson Series

Let $H(t) = H_0 + V(t)$. The interaction-picture perturbation is

$$V_I(t) = e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar}.$$

The evolution operator satisfies $i\hbar\dot{U}_I = V_I U_I$ and expands as

$$U_I = I - \frac{i}{\hbar}\int_0^t V_I(t_1)dt_1 - \frac{1}{\hbar^2}\int_0^t dt_2 \int_0^{t_2} dt_1 V_I(t_2)V_I(t_1) + \cdots .$$

Time ordering cannot be dropped when operators at different times fail to commute.

14.2 Transition Amplitudes and Resonance

For $H_0 |n\rangle = E_n |n\rangle$,

$$c_f^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i(E_f - E_i)t'/\hbar} V_{fi}(t')dt'.$$

A harmonic drive leads to denominators involving detuning $\omega_{fi} \pm \Omega$. As detuning approaches zero,

$$\int_0^t e^{i\delta t'} dt' \longrightarrow t.$$

Such secular terms signal coherent resonant growth and eventual breakdown of a fixed-order approximation.

14.3 Fermi's Golden Rule

For weak coupling into a continuum of final states with density $\rho(E_f)$,

$$\boxed{\Gamma_{i\to f} = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho(E_f)}.$$

The approximate linear law $P \simeq \Gamma t$ pertains to the golden-rule regime, whereas an isolated coherent discrete transition begins quadratically in time.

15 Rabi Oscillations and Raman Processes

15.1 Driven Two-Level System

In an appropriate rotating frame and within the rotating-wave approximation,

$$H_{\text{eff}} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_R \\ \Omega_R & \delta \end{pmatrix}.$$

The transition probability is

$$\boxed{P(t) = \frac{\Omega_R^2}{\Omega_R^2 + \delta^2}\sin^2\left(\frac{\sqrt{\Omega_R^2 + \delta^2}\, t}{2}\right)}.$$

At $\delta = 0$, complete oscillations are possible. The RWA neglects counter-rotating fast terms and is not universally valid for strong drives.

15.2 Three-Level Virtual Transitions

Let $E_1 = E_2 = 0$, $E_3 = \Delta$, set $\hbar = 1$, and use

$$V(t) = \lambda_0 \cos(\Omega t)\big[(|1\rangle + |2\rangle)\langle 3| + \text{h.c.}\big].$$

There is no direct $1 \to 2$ matrix element. The first nonzero pathway is second order through $|3\rangle$:

$$A_{21}^{(2)}(t) = -\lambda_0^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \cos(\Omega t_2)\cos(\Omega t_1) e^{-i\Delta(t_2 - t_1)}.$$

For nonresonant denominators, retain the secular part:

$$\boxed{A_{21}^{(2)}(t) \simeq \frac{i\lambda_0^2 t}{2}\frac{\Delta}{\Delta^2 - \Omega^2}}.$$

Thus $P_{1\to 2} \simeq (\lambda_0^4 t^2/4)\Delta^2/(\Delta^2 - \Omega^2)^2$ while perturbation theory remains valid. Near $\Omega \approx \Delta$, the amplitude behaves as $i\lambda_0^2 t/[4(\Delta - \Omega)]$; exact resonance requires nonperturbative treatment.

15.3 Effective Hamiltonian View

Let P project into a low-energy manifold and $Q = I - P$ into an off-resonant intermediate manifold. Virtual transitions generate schematic second-order couplings

$$H_{\text{eff}}^{(2)} \sim -PVQ(QH_0Q - E)^{-1}QVP.$$

This logic unifies Raman coupling, superexchange, and virtual-state energy shifts.

16 Linear Response and Kubo Formula

16.1 Retarded Susceptibility

For $H'(t) = -f(t)B$ and observable A, the first-order induced response is

$$\delta\langle A(t)\rangle = \int_{-\infty}^t dt'\, \chi_{AB}(t - t')f(t'),$$

$$\boxed{\chi_{AB}(t) = \frac{i}{\hbar}\theta(t)\langle [A_H(t), B_H(0)]\rangle_0}.$$

The step function enforces causality. To derive the expression, use the first-order Dyson expansion of the density operator, substitute $H' = -fB$, and cycle factors under the trace.

16.2 Frequency-Domain Response

With $\chi(\omega) = \int dt\, e^{i\omega t}\chi(t)$,

$$\delta A(\omega) = \chi_{AB}(\omega) f(\omega).$$

The real and imaginary parts describe reactive and absorptive response. Causality connects them through Kramers–Kronig relations. Matrix elements enforce selection rules, while spectral poles encode transition energies.

17 Symmetry and Conservation Laws

17.1 Generators

A continuous transformation is $U(\epsilon) = e^{-i\epsilon G/\hbar}$. If the Hamiltonian is invariant under it, then $[H, G] = 0$ and G is conserved if it has no explicit time dependence. Translation is generated by momentum; spatial rotation by angular momentum.

17.2 Parity and Selection Rules

Parity acts by $P\psi(x) = \psi(-x)$. For an even potential, $[H, P] = 0$ and energy eigenstates can be chosen with definite parity. Since $PxP^{-1} = -x$,

$$\langle n| x |n\rangle = 0 \quad \text{for parity eigenstates.}$$

More generally, an odd operator cannot connect states of the same parity. This explains forbidden transitions and vanishing perturbation matrix elements without integration.

18 Cross-Cutting Ideas

18.1 Small Denominators and Effective Models

Terms of the form $V_{mn}/(E_n - E_m)$ express weak admixture of distant levels. Near degeneracy they become large and require restricted-space diagonalization. The same two-by-two matrix governs flux-ring doublets, spin exchange, coherent driving, and molecular avoided crossings.

18.2 Several Classical Limits

Ehrenfest's theorem describes mean trajectories; coherent states reproduce oscillator classical motion; WKB approximates high-action wavefunctions; stationary phase in the path integral recovers classical equations of motion. These are related but distinct routes to classical physics.

18.3 Real versus Imaginary Time

Unitary real-time evolution $e^{-iHt/\hbar}$ preserves norms and describes physical dynamics. Imaginary-time evolution $e^{-H\tau/\hbar}$ suppresses excited states and, after renormalization, projects toward the lowest-energy component.

19 Master Formula Sheet

19.1 Foundations

$$[x, p] = i\hbar, \quad \langle A\rangle = \langle\psi| A |\psi\rangle, \quad \Delta A \Delta B \geq \tfrac{1}{2}|\langle[A, B]\rangle|.$$

19.2 Evolution

$$U = e^{-iHt/\hbar}, \quad \dot{A} = \tfrac{i}{\hbar}[H, A] + \partial_t A.$$

19.3 Spin

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k, \quad \mathbf{L}\cdot\mathbf{S} = \tfrac{1}{2}(J^2 - L^2 - S^2).$$

19.4 Oscillator

$$H = \hbar\omega(N + \tfrac{1}{2}), \quad [a, a^\dagger] = 1, \quad x = \sqrt{\tfrac{\hbar}{2m\omega}}(a + a^\dagger).$$

19.5 Propagation

$$\partial_t\rho + \nabla\cdot\mathbf{j} = 0, \quad K = \langle x_f| U |x_i\rangle, \quad K = \int \mathcal{D}x\, e^{iS/\hbar}.$$

19.6 Perturbations

$$E_n^{(1)} = V_{nn}, \quad E_n^{(2)} = \sum_{m\neq n} \frac{|V_{mn}|^2}{E_n^{(0)} - E_m^{(0)}}.$$

19.7 Response

$$\chi_{AB}(t) = \frac{i}{\hbar}\theta(t)\langle[A(t), B(0)]\rangle_0 \quad (H' = -fB).$$

20 Problem-Solving Workflow

1. **Choose a representation.** Use the eigenbasis of the dominant Hamiltonian, angular momentum, parity, position, or number states.
2. **Use symmetry early.** Find conserved quantities, degeneracies, and selection rules before computing integrals.
3. **Pick the right method.** Matrix diagonalization, commutators, ladder operators, perturbation theory, WKB, path integrals, or linear response.
4. **Show the essential algebra.** Display matrix elements, denominators, normalization, and chosen approximations.
5. **Check the physics.** Confirm units, limiting cases, probability conservation, and regime of validity.

Common errors: confusing amplitudes with probabilities; changing basis ordering mid-calculation; ignoring phase or complex conjugation; using nondegenerate perturbation theory at degeneracy; dropping time ordering; confusing canonical and mechanical momentum; taking a secular perturbative result to arbitrarily long times.

Recommended References

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